Testing of coarsening mechanisms: Coarsening at random versus subgroup independence

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What's the problem?

Common dealing with incomplete data: assumptions

$\Rightarrow$ Missing at random (MAR) / coarsening at random (CAR)

$\Rightarrow$ Frequently: assumptions only for pragmatic reasons

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What are we doing?

Two types of uninformative coarsening:

The coarsening is independent of ...

- ... the true underlying value (CAR)
- ... the covariate's value (subgroup independence (SI))

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The coarsening is independent of ...

- ... the true underlying value (CAR)
- ... the covariate's value (subgroup independence (SI))

Here: categorical setting

1.) No assumptions about the coarsening
2.) CAR: intuitive insight why not testable
3.) SI: construction of Likelihood Ratio test

Coarse data (categorical setting)

Epistemic interpretation of coarse data (Couso, Dubois, 2014):

LATENT

random sample

$Y_1, \dots, Y_n$

$y_i \in \Omega_Y$

$i = 1, \dots, n$

coarsening

$y_i \in \mathfrak{y}_i$

OBSERVABLE

random sample

$\mathcal{Y}_1, \dots, \mathcal{Y}_n$

$\mathfrak{y}_i \in \mathcal{P}(\Omega_Y) \setminus \{\emptyset\}$

$i = 1, \dots, n$

Coarse data (categorical setting)

Epistemic interpretation of coarse data (Couso, Dubois, 2014):

LATENT		OBSERVABLE
random sample	coarsening $\longrightarrow$	random sample
$Y_1, \ldots, Y_n$		$\mathcal{Y}_1, \ldots, \mathcal{Y}_n$
$y_i \in \Omega_Y$	$y_i \in \mathcal{Y}_i$	$\mathcal{Y}_i \in \mathcal{P}(\Omega_Y) \setminus \{\emptyset\}$
$i = 1, \ldots, n$		$i = 1, \ldots, n$

		$\mathcal{Y}$							
		$\{a\}$	$\{b\}$	$\{c\}$	$\{a,b\}$	$\{a,c\}$	$\{b,c\}$	$\{a,b,c\}$	sum
X	0	$n_{0\{a\}}$	$n_{0\{b\}}$	$n_{0\{c\}}$	$n_{0\{a,b\}}$	$n_{0\{a,c\}}$	$n_{0\{b,c\}}$	$n_{0\{a,b,c\}}$	n_0
	1	$n_{1\{a\}}$	$n_{1\{b\}}$	$n_{1\{c\}}$	$n_{1\{a,b\}}$	$n_{1\{a,c\}}$	$n_{1\{b,c\}}$	$n_{1\{a,b,c\}}$	n_1

Table: Contingency table for coarse data

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		$\mathcal{Y}$			
		$\{a\}$	$\{b\}$	$\{a,b\}$	sum
X	0	$n_{0\{a\}}$	$n_{0\{b\}}$	$n_{0\{a,b\}}$	n_0
	1	$n_{1\{a\}}$	$n_{1\{b\}}$	$n_{1\{a,b\}}$	n_1

Table: Contingency table for missing data

Coarse data (categorical setting)

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LATENT		OBSERVABLE
random sample $Y_1, \ldots, Y_n$	coarsening	random sample $\mathcal{Y}_1, \ldots, \mathcal{Y}_n$
$y_i \in \Omega_Y$, $i = 1, \ldots, n$	$y_i \in \mathfrak{y}_i$	$\mathfrak{y}_i \in \mathcal{P}(\Omega_Y) \setminus \{\emptyset\}$, $i = 1, \ldots, n$

- **UBII**: receipt of Unemployment Benefit II
- $\mathcal{Y}$: categorical income
  - $\{a\}$: < 1000€
  - $\{b\}$: ≥ 1000€
  - $\{a,b\}$: $<$ or $\geq$

		$\mathcal{Y}$			
		{a}	{b}	{a,b}	sum
UBII	0	38	385	95	518
	1	36	42	9	87

Table: PASS, w5 (Trappmann, 2010)

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$i = 1, \ldots, n$

		$\mathcal{Y}$		
		a	b	sum
UBII	0	38+**95**	385	518
	1	36+**9**	42	87

Table: potential true table

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Table: observed table

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UBII	0	38	385+**95**	518
	1	36	42+**9**	87

Table: potential true table

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		{a}	{b}	{a,b}	sum
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		$\mathcal{Y}$		
		a	b	sum
UBII	0	38+**20**	385+**75**	518
	1	36+**5**	42+**4**	87

Table: potential true table

		$\mathcal{Y}$			
		{a}	{b}	{a,b}	sum
UBII	0	38	385	**95**	518
	1	36	42	**9**	87

Table: observed table

LATENT		OBSERVABLE
$\pi_{xy} :=$ $P(Y_i = y \mid X_i = x)$	(error-freeness) **Observation model** $q_{\mathfrak{y}\mid xy} :=$ $P(\mathcal{Y}_i = \mathfrak{y} \mid X_i = x, Y_i = y)$	$p_{x\mathfrak{y}} :=$ $P(\mathcal{Y}_i = \mathfrak{y} \mid X_i = x)$

General ML estimation (Plass, Augustin, Cattaneo, Schollmeyer, 2015)

LATENT

$\vartheta = (\pi_{xy}^{T},\ q_{\mathfrak{y}|xy}^{T})^{T}$

OBSERVABLE

$p_{x\mathfrak{y}} :=$
$P(\mathcal{Y}_i = \mathfrak{y} | X_i = x)$

LATENT	OBSERVABLE	
$\vartheta = (\pi_{xy}^T,\ q_{\mathfrak{y}\mid xy}^T)^T$	$p_{x\mathfrak{y}} := P(\mathcal{Y}_i = \mathfrak{y} \mid X_i = x)$	unique $\arg_{p_{x\mathfrak{y}}} L(p_{x\mathfrak{y}} \mid \text{data}) = \hat{p}_{x\mathfrak{y}} = \frac{n_{x\mathfrak{y}}}{n_x}$

1.) Determine MLE of observed variable distribution

General ML estimation (Plass, Augustin, Cattaneo, Schollmeyer, 2015)

LATENT	OBSERVABLE	
$\vartheta^{(1)}$, $\vartheta^{(2)}$, $\vartheta^{(3)}$, $\dots$	$p_{x\mathfrak{y}} := P(\mathcal{Y}_i = \mathfrak{y} \mid X_i = x)$	unique $\arg_{p_{x\mathfrak{y}}} L(p_{x\mathfrak{y}} \mid \text{data}) = \hat{p}_{x\mathfrak{y}} = \frac{n_{x\mathfrak{y}}}{n_x}$

$$\Phi : [0,1]^{\dim(\Theta_{lat})} \longrightarrow [0,1]^{\dim(\Theta_{obs})}$$

1.) Determine MLE of observed variable distribution

2.) Use connection between both worlds

$$p_{x\mathfrak{y}} = \sum_{y \in \mathfrak{y}} \left(\pi_{xy} \cdot q_{\mathfrak{y}|xy} \right).$$

LATENT

$\vartheta^{(1)}$

$\vartheta^{(2)}$

$\vartheta^{(3)}$

$\dots$

OBSERVABLE

$p_{x\mathfrak{y}} := P(\mathcal{Y}_i = \mathfrak{y} | X_i = x)$

$\arg_{\vartheta^{(1)}} L(\vartheta^{(1)} | \text{data})$

$\arg_{\vartheta^{(2)}} L(\vartheta^{(2)} | \text{data})$

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$\dots$

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3.) Use invariance of the likelihood

$$\hat{\pi}_{xy} \in \left[\frac{n_{x\{y\}}}{n_x}, \frac{\sum_{\mathfrak{y} \ni y} n_{x\mathfrak{y}}}{n_x} \right], \quad \hat{q}_{\mathfrak{y}|xy} \in \left[0, \frac{n_{x\mathfrak{y}}}{n_{x\{y\}} + n_{x\mathfrak{y}}} \right].$$

General ML estimation (Plass, Augustin, Cattaneo, Schollmeyer, 2015)

LATENT		OBSERVABLE	
$\vartheta^{(\mathrm{CAR})}$	$\longrightarrow$	$p_{x\mathfrak{y}} := P(\mathcal{Y}_i = \mathfrak{y} \mid X_i = x)$	unique $\arg_{\vartheta^{(\mathrm{CAR})}} L(\vartheta^{(\mathrm{CAR})} \mid \mathrm{data})$

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Illustrated by the example:

- The probability of $\{a,b\}$ is taken to be independent of the true income category in both subgroups split by UBII:

$$q_{\{a,b\}|0\mathbf{a}} = q_{\{a,b\}|0\mathbf{b}} \quad \text{and} \quad q_{\{a,b\}|1\mathbf{a}} = q_{\{a,b\}|1\mathbf{b}}$$

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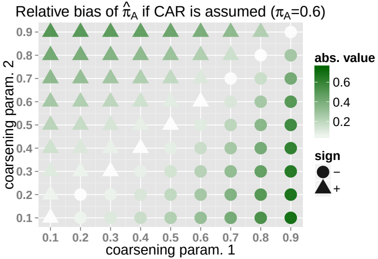

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- Resulting estimators:

$$\hat{\pi}_{xa}^{(CAR)} = \frac{n_{x\{a\}}}{n_{x\{a\}} + n_{x\{b\}}}, \quad \hat{q}_{\{a,b\}|xa}^{(CAR)} = \hat{q}_{\{a,b\}|xb}^{(CAR)} = \frac{n_{x\{a,b\}}}{n_x}$$

Estimators for subgroup 0:

$$\hat{\pi}_{0a} \in [0.07, 0.26], \quad \hat{q}_{\{a,b\}|0a} \in [0,\ 0.71], \quad \hat{q}_{\{a,b\}|0b} \in [0,\ 0.20]$$

$$\hat{\pi}_{0a}^{(CAR)} = 0.09, \qquad \hat{q}_{\{a,b\}|0a}^{(CAR)} = 0.18, \qquad \hat{q}_{\{a,b\}|0b}^{(CAR)} = 0.18$$

Testing CAR? ⇒ :-((e.g. Jaeger, 2006)

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$\vartheta^{(CAR)}$

$$\hat{\pi}^{(CAR)}_{xa} = \frac{n_{x\{a\}}}{n_{x\{a\}} + n_{x\{b\}}}$$

$$\hat{q}^{(CAR)}_{\{a,b\}|xa} = \frac{n_{x\{a,b\}}}{n_x} = \hat{q}_{\{a,b\}|xb}$$

$\underline{\vartheta}$

$$\underline{\hat{\pi}}_{xa} = \frac{n_{x\{a\}}}{n_x}$$

$$\overline{\hat{q}}_{\{a,b\}|xb} = \frac{n_{x\{a,b\}}}{n_{x\{a,b\}} + n_{x\{b\}}}$$

$$\underline{\hat{q}}_{\{a,b\}|xa} = 0$$

$\Big\downarrow$ CAR

$\overline{\vartheta}$

$$\overline{\hat{\pi}}_{xa} = \frac{n_{x\{a\}} + n_{x\{a,b\}}}{n_x}$$

$$\overline{\hat{q}}_{\{a,b\}|xa} = \frac{n_{x\{a,b\}}}{n_{x\{a,b\}} + n_{x\{a\}}}$$

$$\underline{\hat{q}}_{\{a,b\}|xb} = 0$$

⇒ CAR is generally not testable, unless further assumptions about the coarsening are justified

Subgroup independence (SI)

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- Resulting estimators (if well-defined and inside [0,1]):

$$\hat{\pi}^{(SI)}_{xa} = \frac{n_{x\{a\}}}{n_x}\frac{v}{w}, \qquad \hat{q}^{(SI)}_{\{a,b\}|xa} = 1 - \frac{w}{v}, \quad \hat{q}^{(SI)}_{\{a,b\}|xb} = 1 - \frac{w}{z}$$

with $v = n_0 n_{1\{b\}} - n_{0\{b\}} n_1$, $w = n_{0\{a\}} n_{1\{b\}} - n_{0\{b\}} n_{1\{a\}}$ and $z = n_{0A} n_1 - n_{1A} n_0$

Testing SI? ⇒ :-)

Estimators for subgroup 0:

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$$\hat{\pi}_{0a}^{(SI)} = 0.42, \qquad \hat{q}_{\{a,b\}|0a}^{(SI)} = -0.04, \qquad \hat{q}_{\{a,b\}|0b}^{(SI)} = 0.20$$

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$$\hat{\pi}_{0a}^{(SI)} = 0.42, \qquad \hat{q}_{\{a,b\}|0a}^{(SI)} = -0.04, \qquad \hat{q}_{\{a,b\}|0b}^{(SI)} = 0.20$$

⇒ There are data situations that might hint to (partial) incompatibility with SI

⇒ SI is testable in our setting

Testing SI: Hypothesis and test statistic

- Hypothesis:

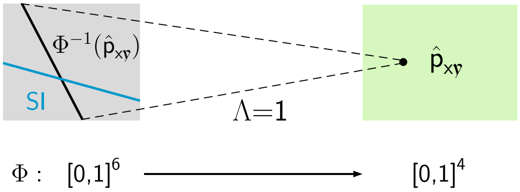
$$\begin{aligned} H_0 &: \quad q_{\mathfrak{y}|0y} = q_{\mathfrak{y}|1y} \text{ for all } y \in \Omega_Y = \{a,b\},\ \mathfrak{y} \in \mathcal{P}(\Omega_Y) \setminus \emptyset \\ H_1 &: \quad q_{\mathfrak{y}|0y} \neq q_{\mathfrak{y}|1y} \text{ for some } y \in \Omega_Y = \{a,b\},\ \mathfrak{y} \in \mathcal{P}(\Omega_Y) \setminus \emptyset. \end{aligned}$$

- Test based on test statistic:

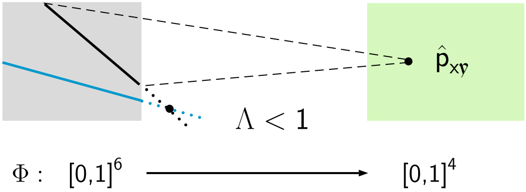
$$\Lambda(\mathfrak{y}_1, \dots, \mathfrak{y}_n, x_1, \dots, x_n) = \frac{\sup_{H_0} L(\vartheta|\mathfrak{y}_1, \dots, \mathfrak{y}_n, x_1, \dots, x_n)}{\sup_{H_0 \cup H_1} L(\vartheta|\mathfrak{y}_1, \dots, \mathfrak{y}_n, x_1, \dots, x_n)},$$

with $\vartheta = (\pi_{0a}, \pi_{1a}, q_{\{a,b\}|0a}, q_{\{a,b\}|1a}, q_{\{a,b\}|0b}, q_{\{a,b\}|1b})^T$

- No evidence to reject SI



- Some evidence to reject SI



- Result of the data example:
  $\Lambda \approx 0.93 \Rightarrow$ slight evidence against SI

Studying more general settings...

Now: number of subgroups m, $|\Omega_Y| = k$

CAR is generally known to be ...

- ... point-identifying
  (Heitjan, Rubin, 1991)
- ... not testable
  (e.g. Jaeger, 2006)

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SI: A determination of the number of degrees of freedom df is crucial:

$$\mathrm{df} = \dim(\Theta_{\mathrm{obs}}) - \dim(\Theta_{\mathrm{lat,\,SI}}) = 2^{\mathrm{m}-1}(2\mathrm{k}-\mathrm{m})-(\mathrm{m}+1)(\mathrm{k}-1)-1$$

- Point-identification and testability are only valid if sufficient subgroups are available inducing $\mathrm{df} \geq 0$
- If $\mathrm{df} \geq 0$, a LR-Test on SI may be constructed with ...
  - ... test statistic $\mathrm{T} = -2 \cdot \log(\Lambda)$ (Wilks, 1938)
  - ... asymptotic distribution of T under H_0
    - $\frac{1}{2}\delta_0 + \frac{1}{2}\chi^2_1$ if df=0
    - χ^2_{df} if df>0,

  where δ_0 is the Dirac distribution at 0 (Chernoff, 1954)

Summary: CAR versus SI

	CAR	SI
Point-identifying?	always	in specific settings
Testability	generally impossible	possible in specific settings

Construction of a hypothesis test:

- H_0: SI, H_1: no SI
- Test statistic based on the Likelihood Ratio

Next step: Generalized version of LR-Test on SI

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