Maxitive Integral of Real-Valued Functions

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- to avoid trivial results, we assume that $0 < \mu(C) < 1$ for some $C \subset \Omega$ (in particular, μ cannot be additive and maxitive at the same time)

extensions of capacities

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- to simplify the results, we consider only extensions F that are:
  - monotonic: $f \leq g \;\Rightarrow\; F(f) \leq F(g)$
  - calibrated: $\alpha \in \mathbb{R} \;\Rightarrow\; F(\alpha\, I_\Omega) = \alpha$
  - null preserving: $\mu\{f \neq 0\} = 0 \;\Rightarrow\; F(f) = 0$

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  - scale invariant: $\beta \in \mathbb{R}_{>0} \ \Rightarrow \ F(\beta f) = \beta F(f)$
  - location invariant: $\alpha \in \mathbb{R} \ \Rightarrow \ F(f + \alpha) = F(f) + \alpha$
  - convex: $\lambda \in (0,1) \ \Rightarrow \ F(\lambda f + (1-\lambda)\, g) \leq \lambda F(f) + (1-\lambda)\, F(g)$

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    i.e., the loss or utility of possible decisions can be measured on an interval scale (Stevens, 1946), since the location of the zero point is of no concern in the decision making
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- when Ω is finite and μ is additive, the integral with respect to μ is a **weighted average**:

$$\int f \,\mathrm{d}\mu = \sum_{\omega \in \Omega} f(\omega)\, \mu\{\omega\}$$

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- when μ is maxitive, its **unique** scale invariant, maxitive extension to $\mathcal{B}^+$ (the set of all bounded functions $f : \Omega \to \mathbb{R}_{\geq 0}$) is the Shilkret integral with respect to μ, which is also convex (Shilkret, 1971):

$$\int^{\mathrm{S}} f \, \mathrm{d}\mu = \bigvee_{x \in \mathbb{R}_{>0}} x \, \mu\{f > x\}$$

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- when Ω is finite and μ is maxitive, the Shilkret integral with respect to μ is a **weighted maximum**:

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- when μ is maxitive, its **unique** location invariant, maxitive extension to $\mathcal{B}$ is the following integral with respect to μ, which is also convex and is therefore called convex integral:

$$\int^{\mathsf{X}} f \, \mathrm{d}\mu = \bigvee_{x \in \mathbb{R} \,:\, \mu\{f > x\} > 0} (x + \mu\{f > x\} - 1)$$

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- when Ω is finite and μ is maxitive, the convex integral with respect to μ is a **penalized maximum**:

$$\int^{\mathsf{X}} f \, \mathrm{d}\mu = \bigvee_{\omega \in \Omega \,:\, \mu\{\omega\} > 0} \left(f(\omega) + \mu\{\omega\} - 1 \right)$$

references

Artzner, P., Delbaen, F., Eber, J.-M., and Heath, D. (1999). Coherent measures of risk. *Math. Finance* 9, 203–228.

Bhaskara Rao, K. P. S., and Bhaskara Rao, M. (1983). *Theory of Charges*. Academic Press.

Cattaneo, M. (2013). Likelihood decision functions. *Electron. J. Stat.* 7, 2924–2946.

de Finetti, B. (1974–1975). *Theory of Probability*. 2 vols. Wiley.

Föllmer, H., and Schied, A. (2011). *Stochastic Finance*. 3rd edn. De Gruyter.

Neyman, J., and Pearson, E. S. (1928). On the use and interpretation of certain test criteria for purposes of statistical inference. *Biometrika* 20A, 175–240 and 263–294.

Savage, L. J. (1972). *The Foundations of Statistics*. 2nd edn. Dover.

Shilkret, N. (1971). Maxitive measure and integration. *Indag. Math.* 33, 109–116.

Stevens, S. S. (1946). On the theory of scales of measurement. *Science* 103, 677–680.