Profile likelihood inference

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example: fundamental problem of practical statistics

- Pearson (1920): An "event" has occurred p times out of $p + q = n$ trials, where we have no a priori knowledge of the frequency of the event in the total population of occurrences. What is the probability of its occurring r times in a further $r + s = m$ trials?

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when $p = 15$, $q = 35$, $n = 50$, $r = 6$, $s = 4$, $m = 10$

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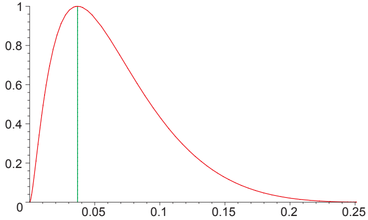

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- **profile likelihood function**: $lik_g : \mathbb{R} \to \mathbb{R}_{\geq 0}$ with

$$lik_g(x) = \sup_{\theta \in \Theta \,:\, g(\theta) = x} lik(\theta)$$

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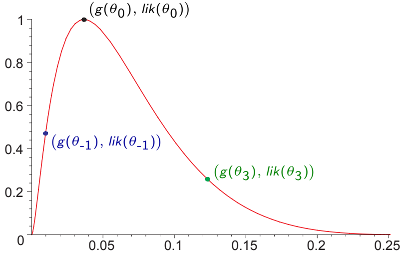

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- in a Bayesian network with categorical variables and known graph, if the dataset is (almost) complete, then the **likelihood function factorizes**:

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$$P\left(C = c \mid F = f,\, (C_1, F_1) = (c_1, f_1),\, \ldots,\, (C_n, F_n) = (c_n, f_n)\right)$$

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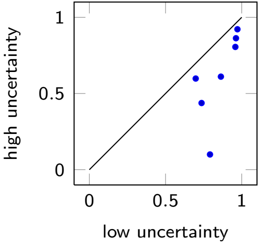
- **example of application:** Bayesian network classifier in which a class is returned only when the probabilities can be estimated with sufficient certainty

- quantity of interest:

$$P\left(C = c \mid F = f,\, (C_1, F_1) = (c_1, f_1),\, \ldots,\, (C_n, F_n) = (c_n, f_n)\right)$$

- experimental results show that the classifier is effective in discriminating "easy" and "hard" instances

accuracy of the classification:



references

- Antonucci, Cattaneo, and Corani (2011). **Likelihood-based naive credal classifier**. In: *ISIPTA '11, Proceedings of the Seventh International Symposium on Imprecise Probability: Theories and Applications*, SIPTA, pp. 21–30.

- Antonucci, Cattaneo, and Corani (2012). **Likelihood-based robust classification with Bayesian networks**. In: *Advances in Computational Intelligence*, Part 3, Springer, pp. 491–500.

- Cattaneo (2010). **Likelihood-based inference for probabilistic graphical models: Some preliminary results**. In: *PGM 2010, Proceedings of the Fifth European Workshop on Probabilistic Graphical Models*, HIIT Publications, pp. 57–64.

- Pearson (1920). **The fundamental problem of practical statistics**. *Biometrika* 13:1–16.