On the implementation of Likelihood-based Imprecise Regression

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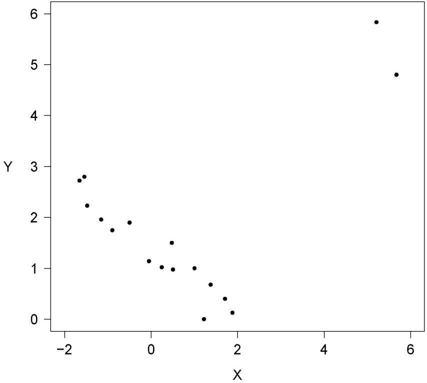
15 December 2011

simple linear regression



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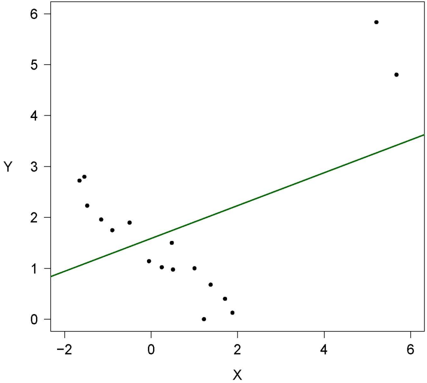

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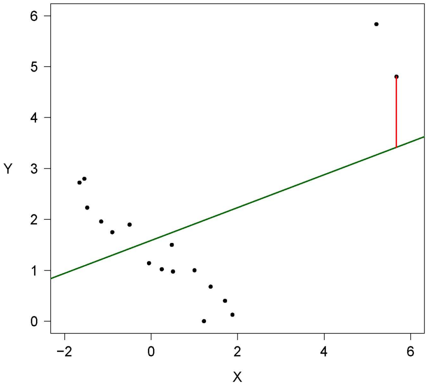

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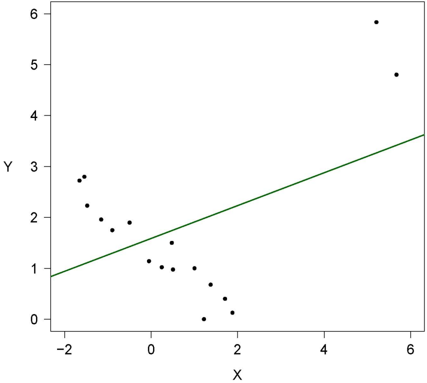

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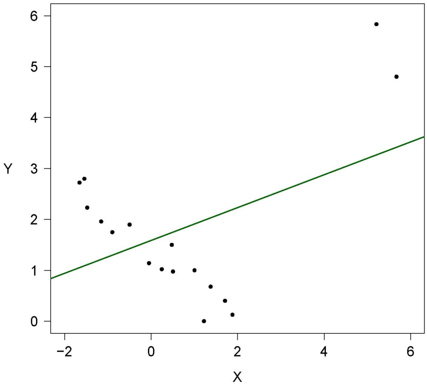

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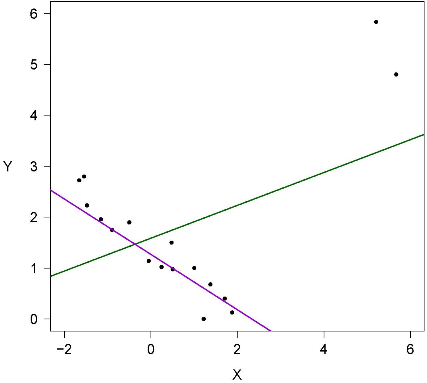

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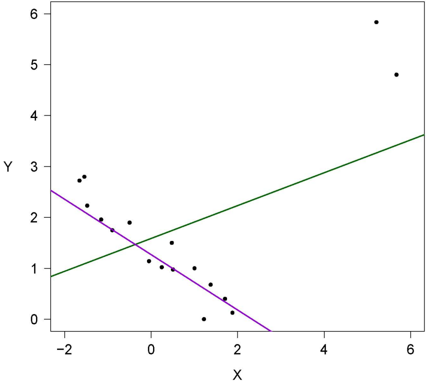

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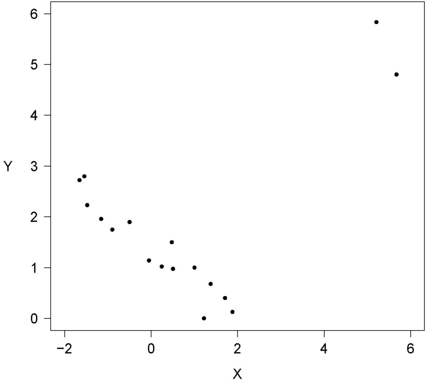
imprecisely observed data



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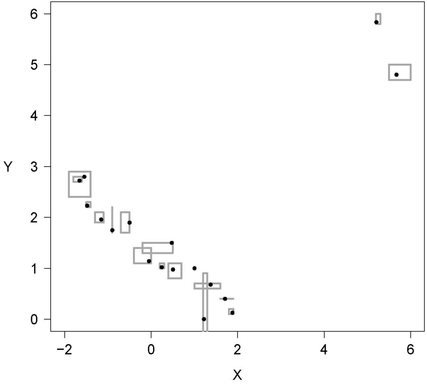

- imprecise data: $(\underline{x}_i, \overline{x}_i, \underline{y}_i, \overline{y}_i) \in \overline{\mathbb{R}}^4$ for each $i \in \{1, \ldots, n\}$

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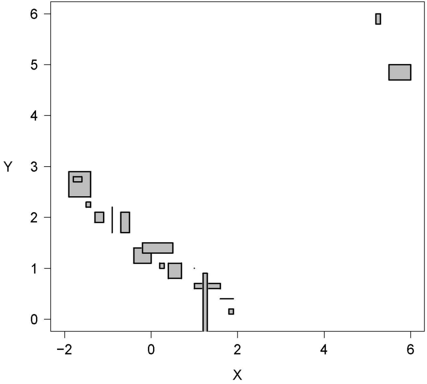

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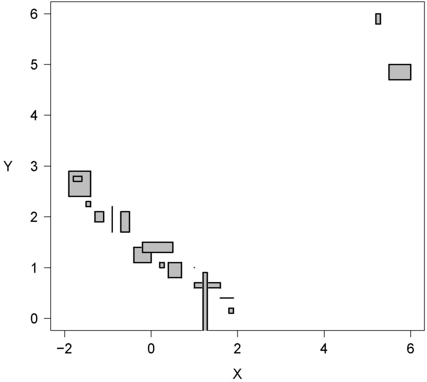

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$$\underline{X}_i \leq X_i \leq \overline{X}_i \quad \text{and} \quad \underline{Y}_i \leq Y_i \leq \overline{Y}_i \quad P\text{-a.s.}$$

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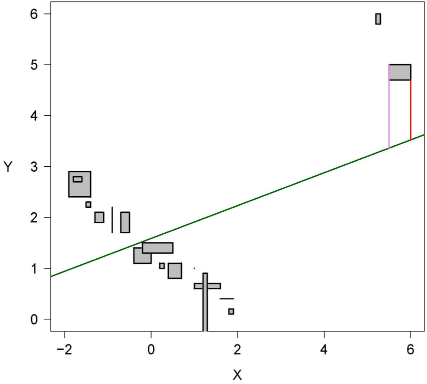

imprecise residuals:

$$\underline{r}_{f,i} = \min_{(x,y)\in[\underline{x}_i,\overline{x}_i]\times[\underline{y}_i,\overline{y}_i]} |y - f(x)|$$

$$\overline{r}_{f,i} = \sup_{(x,y)\in[\underline{x}_i,\overline{x}_i]\times[\underline{y}_i,\overline{y}_i]} |y - f(x)|$$

for each $f \in \mathcal{F}$, $i \in \{1, \ldots, n\}$

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- likelihood function: $lik : P \mapsto \prod_{i=1}^{n} \dfrac{P(\underline{X}_i = \underline{x}_i,\, \overline{X}_i = \overline{x}_i,\, \underline{Y}_i = \underline{y}_i,\, \overline{Y}_i = \overline{y}_i)}{\widehat{P}_{\underline{X},\overline{X},\underline{Y},\overline{Y}}(\underline{x}_i, \overline{x}_i, \underline{y}_i, \overline{y}_i)}$

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- likelihood-based learning of imprecise probability model: $\mathcal{P}_{>\beta} = \{P \in \mathcal{P} : lik(P) > \beta\}$ for some cutoff point $\beta \in (0, 1)$

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$$\mathcal{C}_f = [\underline{r}_{f,(n-\overline{k}+1)},\, \overline{r}_{f,(\overline{k})}],$$

where $\sqrt[n]{\beta} \mapsto \frac{\overline{k}}{n}$ is a decreasing bijection $[\frac{1}{2}, 1) \to (\frac{1}{2}, 1]$

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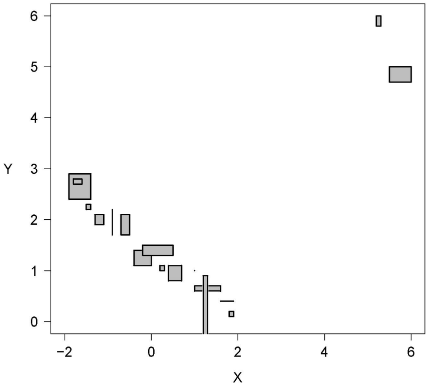
- interval dominance: $\mathcal{U} = \{f \in \mathcal{F} : \underline{r}_{f,(n-\overline{k}+1)} \leq \overline{r}_{f_{LRM},(\overline{k})}\}$ is the set of all undominated regression lines

algorithm for f_{LRM}



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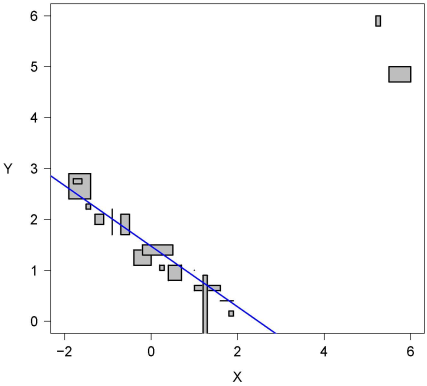

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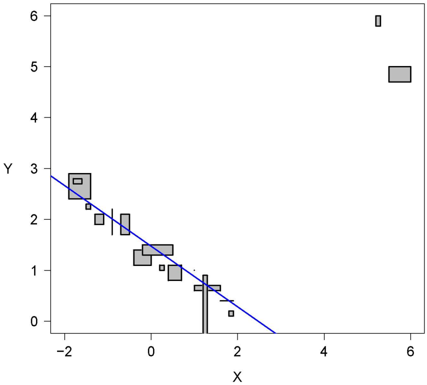

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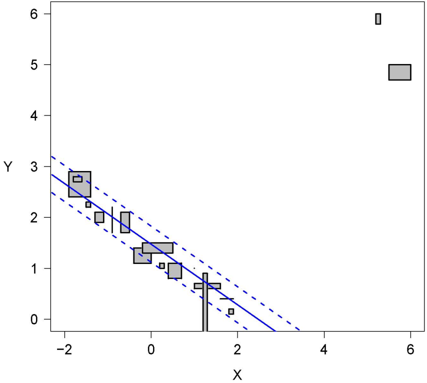

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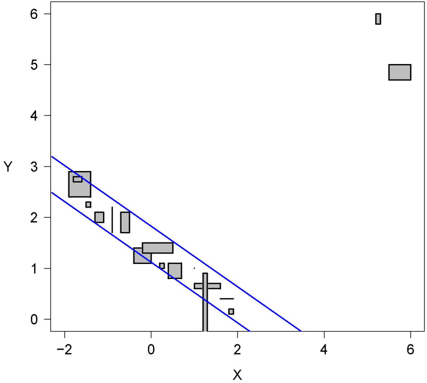

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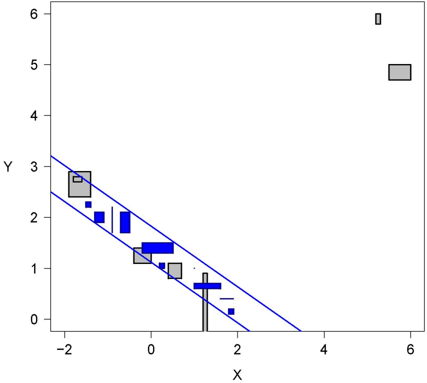

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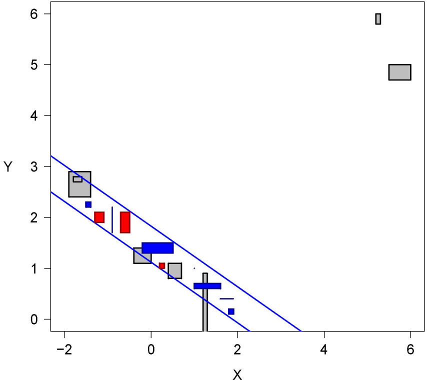

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- if the slope $b_{LRM} \neq 0$, then the imprecise data contained in $f_{LRM} \pm \overline{r}_{f_{LRM},(\overline{k})}$ are bounded and (at least) 3 of them touch the boundary of the strip

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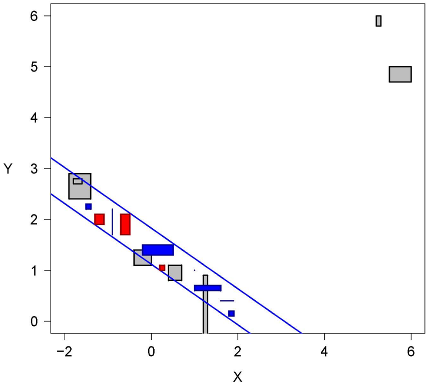

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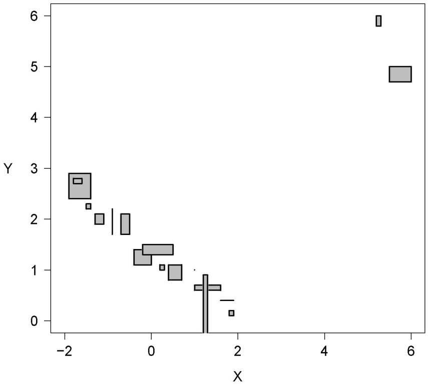
- if the slope $b_{LRM} \neq 0$, then the imprecise data contained in $f_{LRM} \pm \overline{r}_{f_{LRM},(\overline{k})}$ are bounded and (at least) 3 of them touch the boundary of the strip
- therefore, b_{LRM} is either 0 or it is determined by a couple of bounded imprecise data, which gives us at most $4\binom{n}{2} + 1$ possible values for b_{LRM}

undominated regression lines



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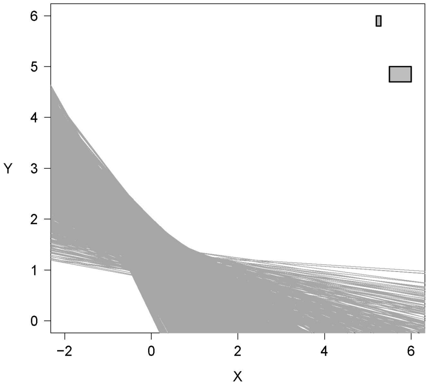

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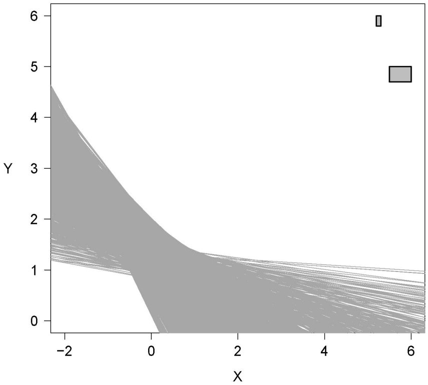
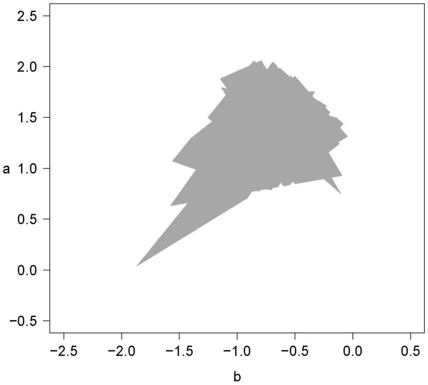
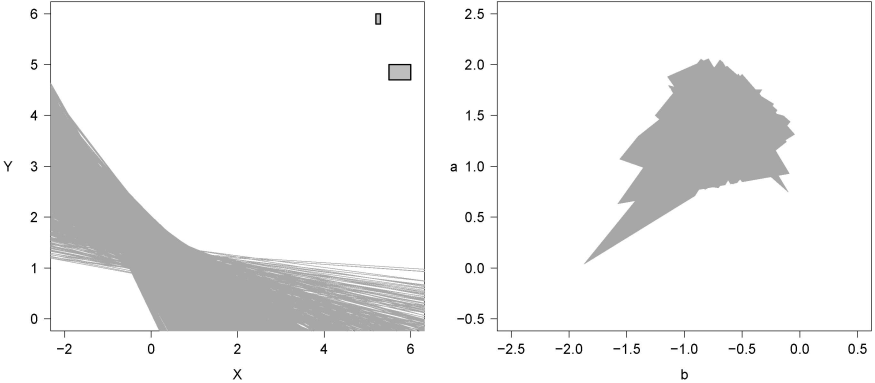




- set of undominated parameters: $\left\{(a, b) \in \mathbb{R}^2 : f_{a,b} \in \mathcal{U}\right\}$

undominated regression lines



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$$= \bigcup_{i=1}^{\overline{k}} \left\{(a,b) \in \mathbb{R}^2 : \underline{d}_{b,(i+n-\overline{k})} - \overline{r}_{f_{LRM},(\overline{k})} \leq a \leq \overline{d}_{b,(i)} + \overline{r}_{f_{LRM},(\overline{k})}\right\},$$

where $\underline{d}_{b,i} = \inf_{x \in [\underline{x}_i, \overline{x}_i]}(\underline{y}_i - b\,x)$ and $\overline{d}_{b,i} = \sup_{x \in [\underline{x}_i, \overline{x}_i]}(\overline{y}_i - b\,x)$

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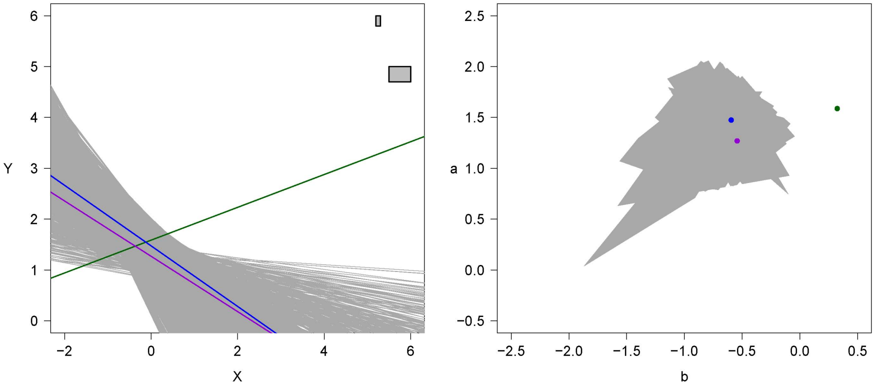

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- example: $\mathcal{C}_{f_{LRM}} = [0,\ 0.354]$, $\mathcal{C}_{f_{LMS}} = [0.002,\ 0.442]$, $\mathcal{C}_{f_{LS}} = [0.909,\ 1.502]$

statistical properties of LIR

- breakdown point: $\varepsilon^*_{LIR} = 1 - \frac{\overline{k}}{n} \xrightarrow{n \to \infty} \frac{1}{2}$

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β	n	$\underline{P}(med\, R_{f,i} \in \mathcal{C}_f)$	F_0	$\underline{P}(f_{a_0,b_0} \in \mathcal{U})$
0.5	20	0.737	*Normal*	0.83
			Cauchy	0.97
	1000	0.758	*Normal*	1.00
			Cauchy	1.00
0.75	20	0.497	*Normal*	0.39
			Cauchy	0.72
	1000	0.533	*Normal*	0.91
			Cauchy	1.00
0.999	20	0.176	*Normal*	0.03
			Cauchy	0.11
	1000	0.025	*Normal*	0.00
			Cauchy	0.01

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