Regression with Imprecise Data:

A Robust Approach

Marco Cattaneo and Andrea Wiencierz

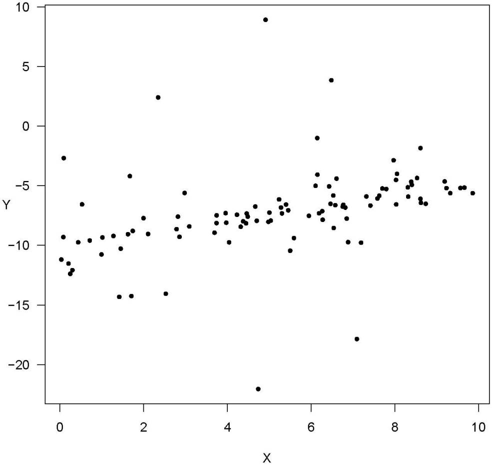
Department of Statistics, LMU Munich

July 25, 2011

ISIPTA '11, Innsbruck, Austria

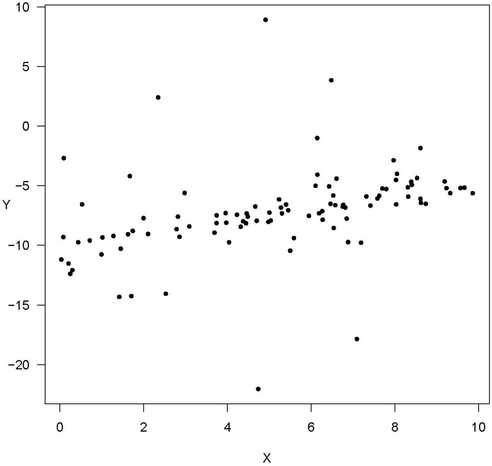
Regression Analysis

- Consider data on two variables, X and Y.



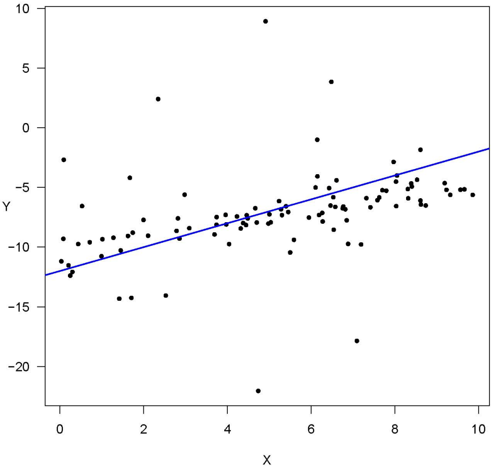
Regression Analysis

- Consider data on two variables, X and Y.
- The aim is to investigate the relationship between X and Y.



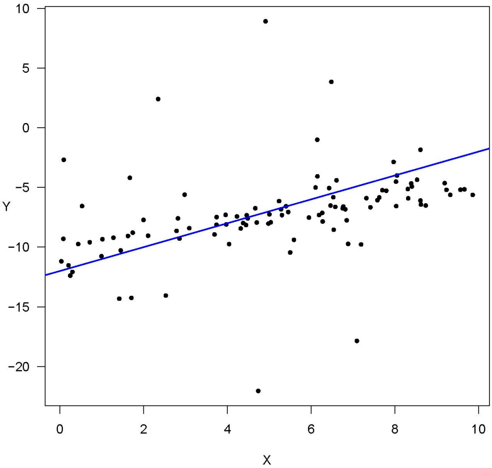
Linear Regression 1

- The relationship between X and Y is described by: $Y = f(X) = a + b\,X$, $a, b \in \mathbb{R}$.



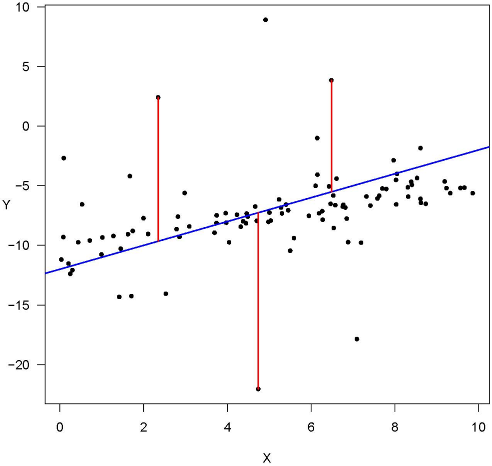
Linear Regression 1

- The relationship between X and Y is described by: $Y = f(X) = a + b\,X$, $a, b \in \mathbb{R}$.
- For which a and b does the function f best fit the data?



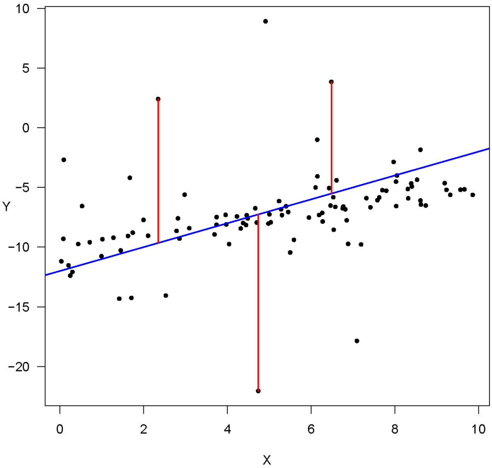
Linear Regression 2

- For a given f, the (absolute) residuals are defined by $R_{f,i} := |Y_i - f(X_i)|$.



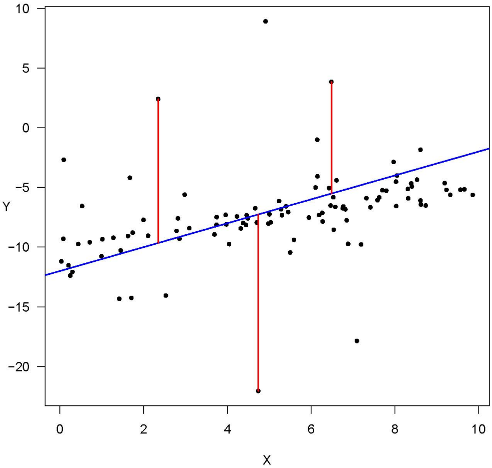
Linear Regression 2

- For a given f, the (absolute) residuals are defined by $R_{f,i} := |Y_i - f(X_i)|$.
- Ordinary Least Squares: f_{OLS} minimizes the mean of ${R_{f,i}}^2$.



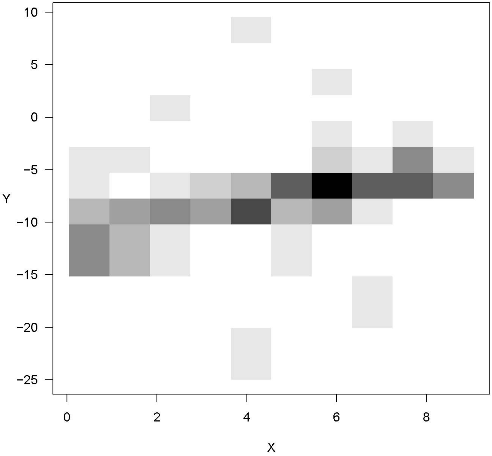
Linear Regression 2

- For a given f, the (absolute) residuals are defined by $R_{f,i} := |Y_i - f(X_i)|$.
- Ordinary Least Squares: f_{OLS} minimizes the mean of ${R_{f,i}}^2$.
- Least Median of Squares: f_{LMS} minimizes the median of ${R_{f,i}}^2$.



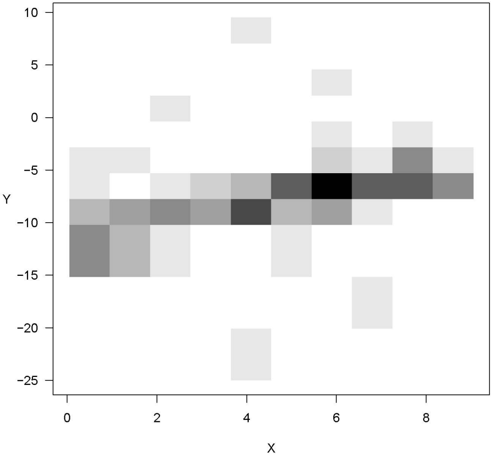
Imprecise Data

- Observation spaces of X and Y are partitioned into disjoint intervals.



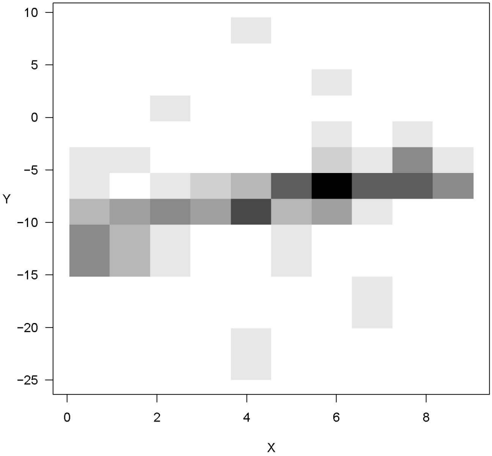
Imprecise Data

- Observation spaces of X and Y are partitioned into disjoint intervals.
- Rectangular data: $[\underline{X}_i, \overline{X}_i) \times [\underline{Y}_i, \overline{Y}_i)$.



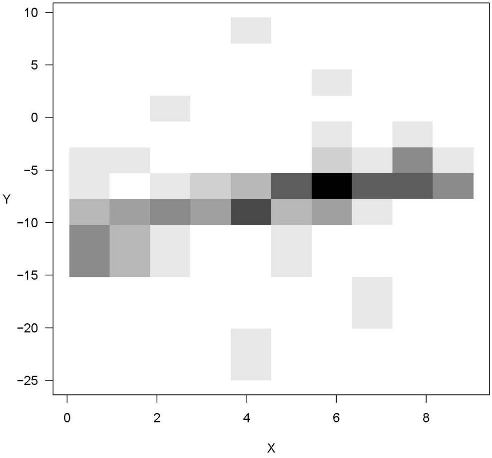
Imprecise Data

- Observation spaces of X and Y are partitioned into disjoint intervals.
- Rectangular data: $[\underline{X}_i, \overline{X}_i) \times [\underline{Y}_i, \overline{Y}_i)$.
- How to draw a line that reflects the relationship between X and Y?



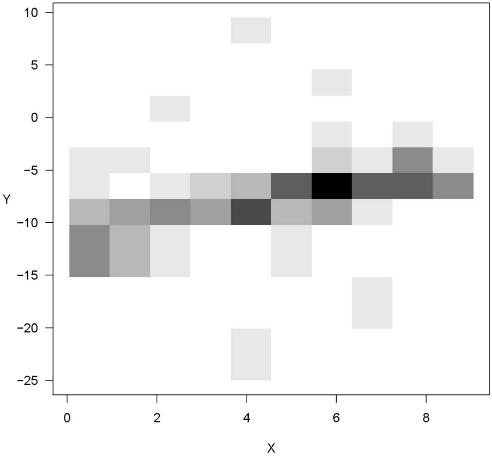
Imprecise Data

- Observation spaces of X and Y are partitioned into disjoint intervals.
- Rectangular data: $[\underline{X}_i, \overline{X}_i) \times [\underline{Y}_i, \overline{Y}_i)$.
- How to draw a line that reflects the relationship between X and Y?
- Common simple method: OLS based on interval midpoints.



Imprecise Data

- Observation spaces of X and Y are partitioned into disjoint intervals.
- Rectangular data: $[\underline{X}_i, \overline{X}_i) \times [\underline{Y}_i, \overline{Y}_i)$.
- How to draw a line that reflects the relationship between X and Y?
- Common simple method: OLS based on interval midpoints.
- Further approaches: e.g. Domingues et al. (2010) or Ferson et al. (2007).



A Robust Approach to Regression with Imprecise Data

- Theoretical framework: likelihood-based decisions (Cattaneo, 2007).

A Robust Approach to Regression with Imprecise Data

- Theoretical framework: likelihood-based decisions (Cattaneo, 2007).
- We assume that the variables have precise values, which are imprecisely observed:

$$V_i := (X_i, Y_i) \quad \text{and} \quad V_i^* := [\underline{X}_i, \overline{X}_i) \times [\underline{Y}_i, \overline{Y}_i), \quad i = 1, \ldots, n.$$

A Robust Approach to Regression with Imprecise Data

- Theoretical framework: likelihood-based decisions (Cattaneo, 2007).
- We assume that the variables have precise values, which are imprecisely observed:

$$V_i := (X_i, Y_i) \quad \text{and} \quad V_i^* := [\underline{X}_i, \overline{X}_i) \times [\underline{Y}_i, \overline{Y}_i), \quad i = 1, \ldots, n.$$

- Nonparametric probability model:

$$\mathcal{P} := \{P : \ (V_i, V_i^*),\ i = 1, \ldots, n,\ \text{i.i.d.} \ \wedge \ P(V_i \in V_i^*) \geq 1 - \varepsilon\},$$

for some $\varepsilon \in [0, 1]$.

A Robust Approach to Regression with Imprecise Data

- Theoretical framework: likelihood-based decisions (Cattaneo, 2007).
- We assume that the variables have precise values, which are imprecisely observed:

$$V_i := (X_i, Y_i) \quad \text{and} \quad V_i^* := [\underline{X}_i, \overline{X}_i) \times [\underline{Y}_i, \overline{Y}_i), \quad i = 1, \dots, n.$$

- Nonparametric probability model:

$$\mathcal{P} := \left\{ P : \ (V_i, V_i^*),\ i = 1, \dots, n,\ \text{i.i.d.} \ \wedge \ P(V_i \in V_i^*) \geq 1 - \varepsilon \right\},$$

for some $\varepsilon \in [0, 1]$.

- Given $V_1^*, \dots, V_n^*$, we reduce $\mathcal{P}$ via the likelihood function to the set $\mathcal{P}_{>\beta} := \left\{ P \in \mathcal{P} : \ lik(P) > \beta \right\}$, for some (chosen) $\beta \in (0, 1)$.

A Robust Approach to Regression with Imprecise Data

- Theoretical framework: likelihood-based decisions (Cattaneo, 2007).
- We assume that the variables have precise values, which are imprecisely observed:

$$V_i := (X_i, Y_i) \quad \text{and} \quad V_i^* := [\underline{X}_i, \overline{X}_i) \times [\underline{Y}_i, \overline{Y}_i), \quad i = 1, \dots, n.$$

- Nonparametric probability model:

$$\mathcal{P} := \{P : \ (V_i, V_i^*), \ i = 1, \dots, n, \text{ i.i.d.} \ \wedge \ P(V_i \in V_i^*) \geq 1 - \varepsilon\},$$

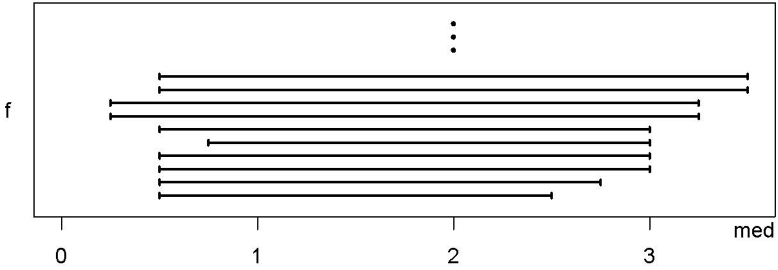
for some $\varepsilon \in [0, 1]$.

- Given $V_1^*, \dots, V_n^*$, we reduce $\mathcal{P}$ via the likelihood function to the set

$\mathcal{P}_{>\beta} := \{P \in \mathcal{P} : \ lik(P) > \beta\}$, for some (chosen) $\beta \in (0, 1)$.

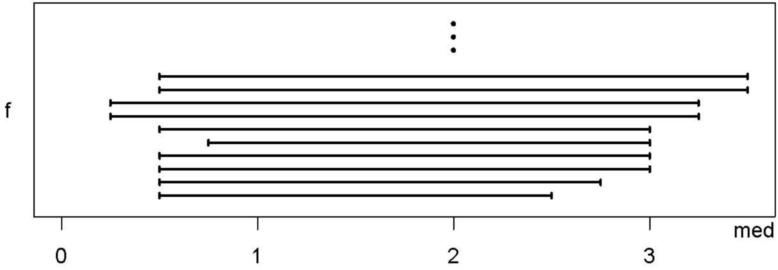
- The set $\mathcal{P}_{>\beta}$ determines interval-valued estimates of the median of the (absolute) residuals $R_{f,i}$ for the regression functions f.

Regression as a Decision Problem



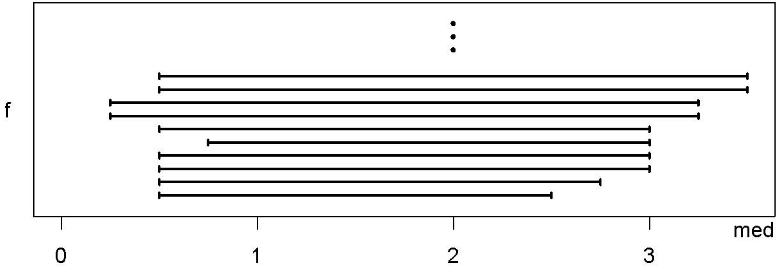
- For each regression function f, we have an interval-valued evaluation, which is a confidence interval for the median of $R_{f,i}$.

Regression as a Decision Problem



- For each regression function f, we have an interval-valued evaluation, which is a confidence interval for the median of $R_{f,i}$.
- Interval dominance leads to a set of optimal regression functions.

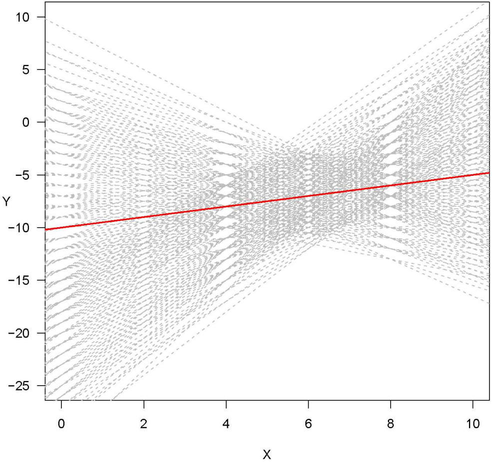
Regression as a Decision Problem



- For each regression function f, we have an interval-valued evaluation, which is a confidence interval for the median of $R_{f,i}$.
- Interval dominance leads to a set of optimal regression functions.
- (Γ-)minimax leads to one optimal regression function.

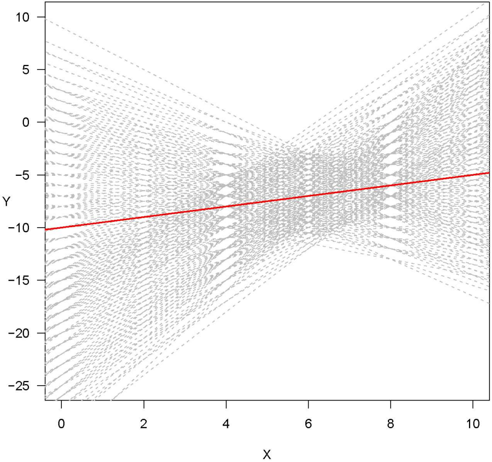
Result of Regression 1

- Regression analysis of the imprecise dataset shown before.



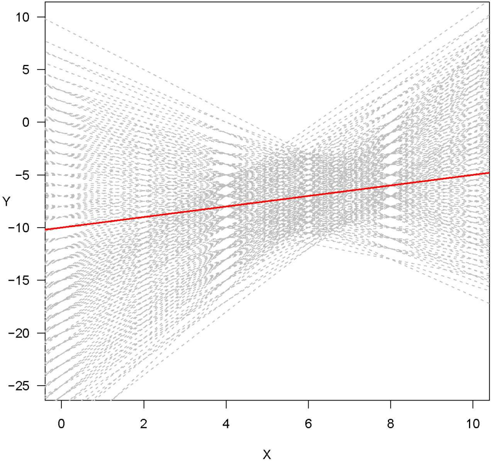
Result of Regression 1

- Regression analysis of the imprecise dataset shown before.
- We considered linear regression functions, $f(X) = a + bX$.



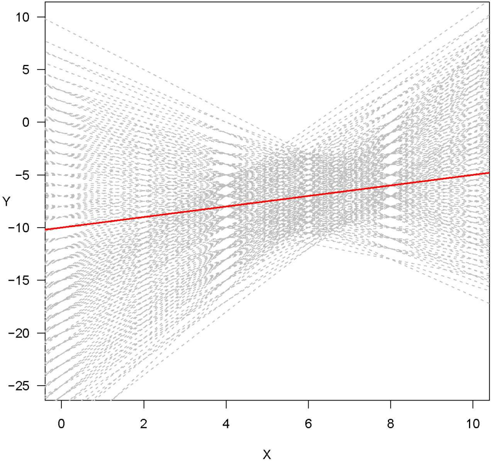
Result of Regression 1

- Regression analysis of the imprecise dataset shown before.
- We considered linear regression functions, $f(X) = a + bX$.
- Calculations are based on a grid search.



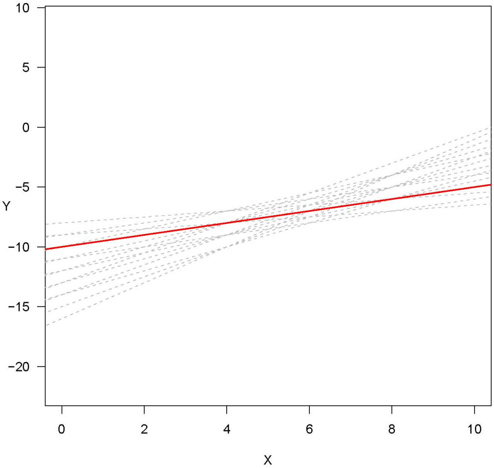
Result of Regression 1

- Regression analysis of the imprecise dataset shown before.
- We considered linear regression functions, $f(X) = a + b\,X$.
- Calculations are based on a grid search.
- The imprecision of the result mainly reflects the imprecision of the data.



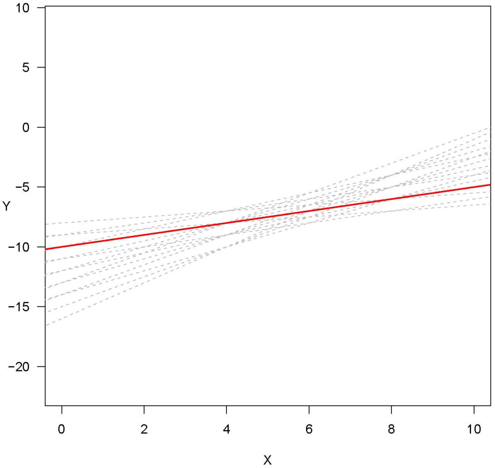
Result of Regression 2

- We performed the same analysis on the dataset with imprecise observations of X, but precise data of Y.



Result of Regression 2

- We performed the same analysis on the dataset with imprecise observations of X, but precise data of Y.
- The result of the regression analysis is much more precise.



Summary and Outlook

- We introduced a likelihood-based imprecise regression approach.

Summary and Outlook

- We introduced a likelihood-based imprecise regression approach.
- The approach is very general and the regression method covers many different settings:

Summary and Outlook

- We introduced a likelihood-based imprecise regression approach.
- The approach is very general and the regression method covers many different settings:
  - We can consider all kinds of imprecise data, not only disjoint intervals.

Summary and Outlook

- We introduced a likelihood-based imprecise regression approach.
- The approach is very general and the regression method covers many different settings:
  - We can consider all kinds of imprecise data, not only disjoint intervals.
  - The imprecise data can be wrong with a certain probability ($\varepsilon > 0$).

Summary and Outlook

- We introduced a likelihood-based imprecise regression approach.
- The approach is very general and the regression method covers many different settings:
  - We can consider all kinds of imprecise data, not only disjoint intervals.
  - The imprecise data can be wrong with a certain probability ($\varepsilon > 0$).
  - It is possible to consider more than one explanatory variable.

Summary and Outlook

- We introduced a likelihood-based imprecise regression approach.
- The approach is very general and the regression method covers many different settings:
  - We can consider all kinds of imprecise data, not only disjoint intervals.
  - The imprecise data can be wrong with a certain probability ($\varepsilon > 0$).
  - It is possible to consider more than one explanatory variable.
  - There can be imprecision in dependent and explanatory variables at the same time.

Summary and Outlook

- We introduced a likelihood-based imprecise regression approach.
- The approach is very general and the regression method covers many different settings:
  - We can consider all kinds of imprecise data, not only disjoint intervals.
  - The imprecise data can be wrong with a certain probability ($\varepsilon > 0$).
  - It is possible to consider more than one explanatory variable.
  - There can be imprecision in dependent and explanatory variables at the same time.
  - We can consider arbitrary regression functions, not only linear ones.

Summary and Outlook

- We introduced a likelihood-based imprecise regression approach.
- The approach is very general and the regression method covers many different settings:
  - We can consider all kinds of imprecise data, not only disjoint intervals.
  - The imprecise data can be wrong with a certain probability ($\varepsilon > 0$).
  - It is possible to consider more than one explanatory variable.
  - There can be imprecision in dependent and explanatory variables at the same time.
  - We can consider arbitrary regression functions, not only linear ones.
  - Instead of the median we can use any other quantile.

Summary and Outlook

- We introduced a likelihood-based imprecise regression approach.
- The approach is very general and the regression method covers many different settings:
  - We can consider all kinds of imprecise data, not only disjoint intervals.
  - The imprecise data can be wrong with a certain probability ($\varepsilon > 0$).
  - It is possible to consider more than one explanatory variable.
  - There can be imprecision in dependent and explanatory variables at the same time.
  - We can consider arbitrary regression functions, not only linear ones.
  - Instead of the median we can use any other quantile.
- The presented regression method yields very robust results.

Cattaneo, M. (2007). *Statistical Decisions Based Directly on the Likelihood Function*. PhD thesis, ETH Zurich. `doi:10.3929/ethz-a-005463829`.

Domingues, M. A. O., de Souza, R. M. C. R., and Cysneiros, F. J. A. (2010). A robust method for linear regression of symbolic interval data. *Pattern Recognit. Lett.* 31, 1991–1996.

Ferson, S., Kreinovich, V., Hajagos, J., Oberkampf, W., and Ginzburg, L. (2007). *Experimental Uncertainty Estimation and Statistics for Data Having Interval Uncertainty*. Technical Report SAND2007-0939. Sandia National Laboratories.