Support Vector Regression with interval data

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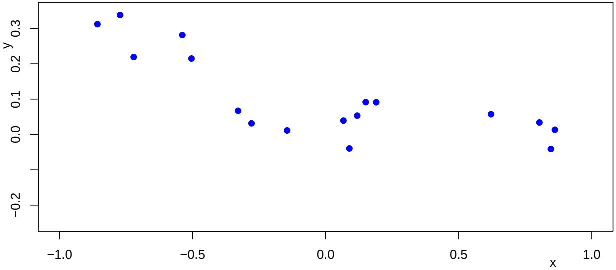
support vector regression (SVR)



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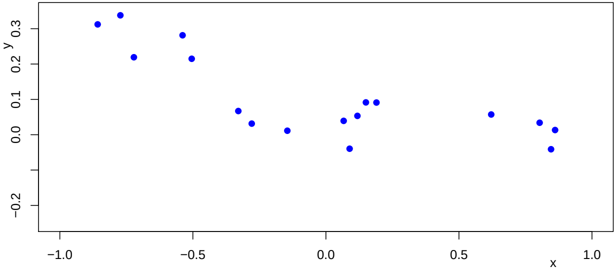

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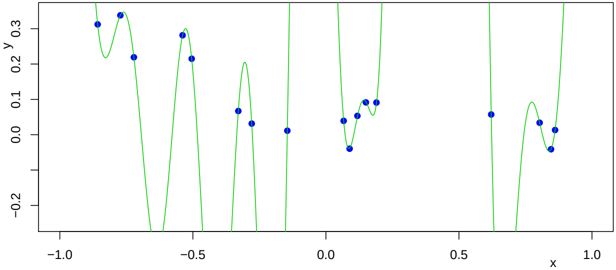

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- regression function: $f \in \mathcal{F}$ minimizing

$$\mathcal{E}(f) = \frac{1}{n} \sum_{i=1}^{n} \psi\left(|y_i - f(x_i)|\right),$$

where $\psi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is convex with $\psi(0) = 0$, e.g., linear loss $\psi : r \mapsto r$

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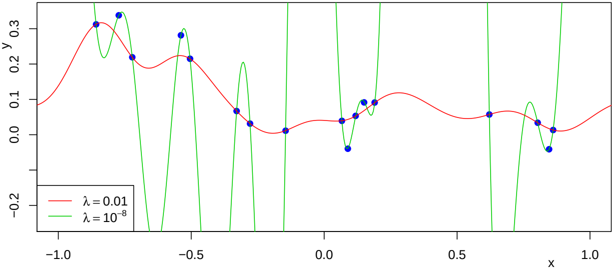

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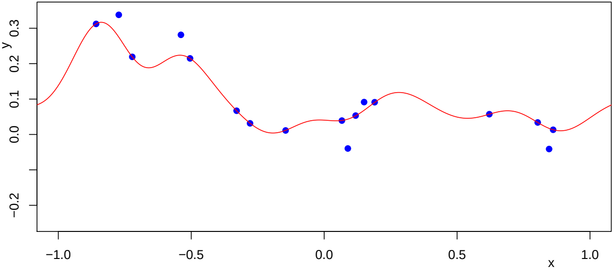
representer theorem (RT)



- the regression function minimizing $\mathcal{E}(f)$ exists, is unique, and has the form

$$f = \sum_{j=1}^{n} \alpha_j \, \kappa(\,\cdot\,, x_j),$$

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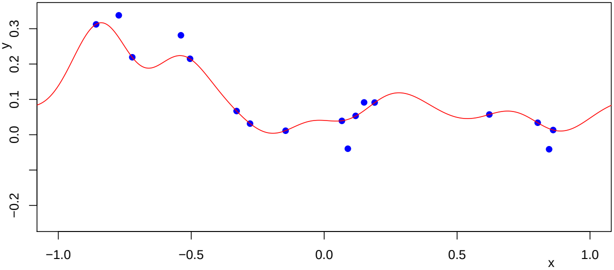

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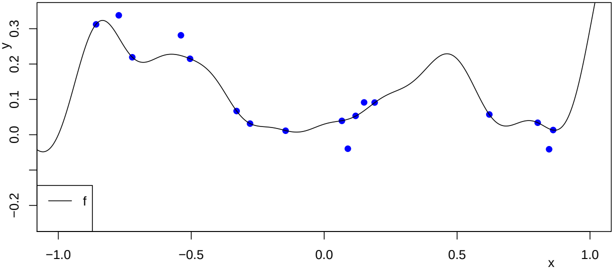

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- core of the proof: the structure of $\mathcal{F}$ implies that for each f, the orthogonal projection $f' = \sum_{j=1}^{n} \alpha'_j \, \kappa(\,\cdot\,, x_j)$ of f on the subspace spanned by the functions $\kappa(\,\cdot\,, x_j)$ satisfies $f'(x_i) = f(x_i)$, and therefore $\mathcal{E}(f') \leq \mathcal{E}(f)$

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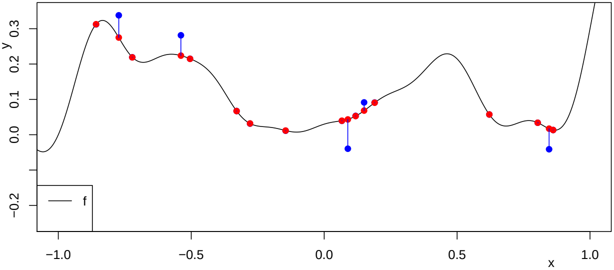

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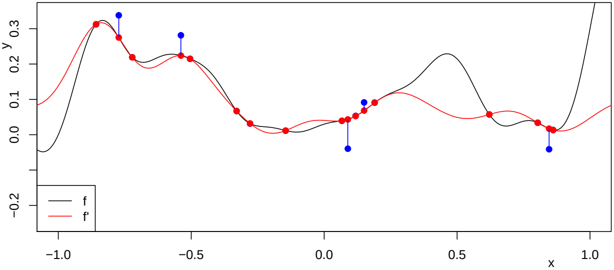

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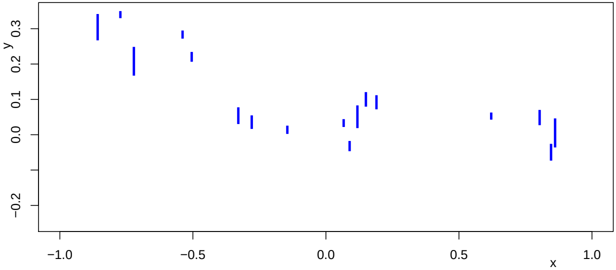

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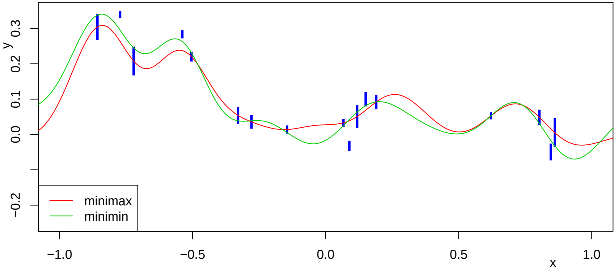

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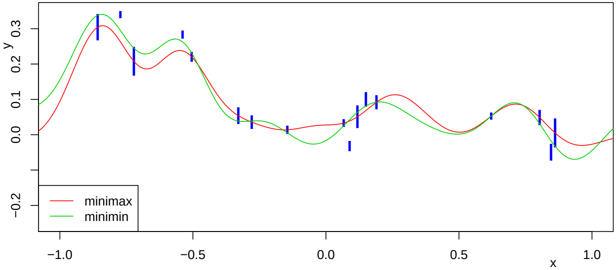

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- RT: the regression functions minimizing $\underline{\mathcal{E}}(f)$ and $\overline{\mathcal{E}}(f)$ exist, are unique, and have the form $f = \sum_{j=1}^{n} \alpha_j \, \kappa(\,\cdot\,, x_j)$, so that the minimizations of $\underline{\mathcal{E}}(f)$ and $\overline{\mathcal{E}}(f)$ become convex optimization tasks in n variables $\alpha_1, \ldots, \alpha_n$

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  are minimized:
  - a regression function minimizing $\underline{\mathcal{E}}(f)$ would have the form $f = \sum_{j=1}^n \alpha_j \, \kappa(\,\cdot\,, x_j)$, where $\alpha_j \in \mathbb{R}$ and $x_j \in [\underline{x}_j, \overline{x}_j]$, but in general $\underline{\mathcal{E}}$ is not convex

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  - by contrast, $\overline{\mathcal{E}}$ is convex, but a regression function minimizing $\overline{\mathcal{E}}(f)$ does not necessarily have the form $f = \sum_{j=1}^{n} \alpha_j \, \kappa(\,\cdot\,, x_j)$, where $\alpha_j \in \mathbb{R}$ and $x_j \in [\underline{x}_j, \overline{x}_j]$

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- the more general case with interval data $[\underline{x}_i, \overline{x}_i] \times [\underline{y}_i, \overline{y}_i] \subset \mathbb{R}^d \times \mathbb{R}$ also presents the above difficulties