The likelihood approach to statistics as a theory of imprecise probability

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- $\log \frac{lik(P_1)}{lik(P_2)}$ is the *information for discrimination* (or *weight of evidence*) in favor of P_1 against P_2
- in particular, a constant *lik* describes the case of **no information** for discrimination among the probabilistic models in $\mathcal{P}$

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- the penalty term in penalized likelihood methods can often be interpreted as a prior *lik*

- the choice of a prior *lik* seems better supported by intuition than the choice of a prior probability measure: in particular, a constant *lik* describes the case of no information (**complete ignorance**)

imprecise probability

- the uncertain knowledge about the value $g(P)$ of a function $g : \mathcal{P} \to \mathcal{G}$ is described by the **profile** likelihood function

$$lik_g(\gamma) \propto \sup_{P \in \mathcal{P} \,:\, g(P) = \gamma} lik(P) \quad \text{on } \mathcal{G}$$

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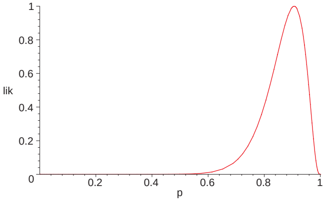
- example: profile likelihood function for the probability p of observing at least 3 successes in the next 5 experiments (Bernoulli trials), after having observed 38 successes in 50 experiments



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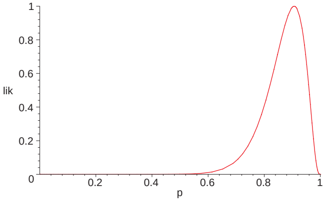

- *normalized* likelihood functions are a possible interpretation of membership functions of fuzzy sets: in this sense, the hierarchical model is a **fuzzy probability** measure, and the above graph shows the membership function of a fuzzy probability value

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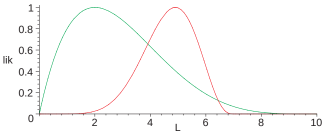

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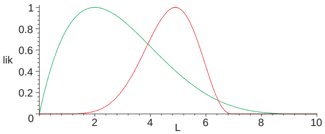

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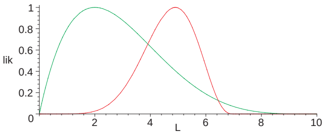

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- the only likelihood-based decision criterion satisfying some basic properties is the **MPL criterion** with $\alpha \in (0, \infty)$:

$$\text{minimize} \quad \sup_{P \in \mathcal{P}} lik(P)^{\alpha} \, L(P, d)$$

comparison of hierarchical and Bayesian models

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    (*many bad probabilistic models make a good one*)

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  - the precise Bayesian model: the ability to describe the state of **complete ignorance**
  - the imprecise Bayesian model: the ability to **get out** of the state of complete ignorance