A statistician's view on fuzzy sets

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kinds of uncertainty

probability distributions

$X \sim F$, e.g.: $X \sim \mathcal{N}(0, 1)$

sets

$x \in A$, e.g.: $x \in [-10, 10]$

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classical approach to statistics: theory of interval probability:	probability distributions ($X \sim F_\theta$) and sets ($\theta \in \Theta$)

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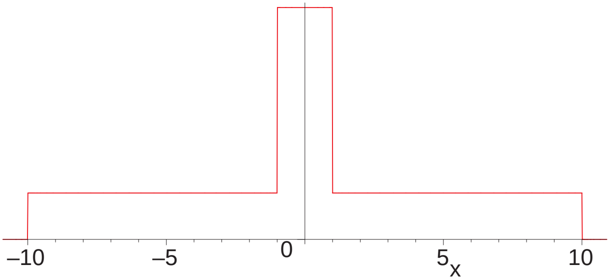
classical approach to statistics: theory of interval probability:	probability distributions ($X \sim F_\theta$) and sets ($\theta \in \Theta$)
theory of fuzzy probability:	probability distributions ($X \sim F_\theta$) and fuzzy sets ($\theta \in F$, $\theta \sim \mu_F$)

fuzzy sets vs (subjective) probability distributions



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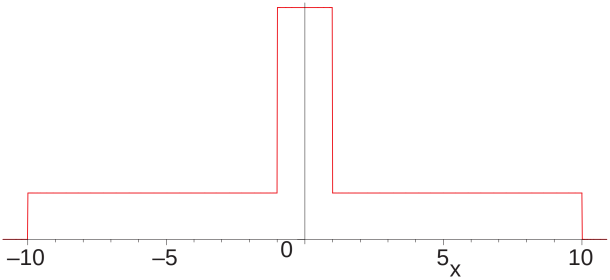

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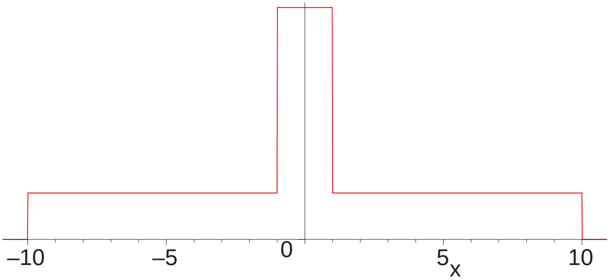

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(by contrast, the rules of probability theory do not depend on the interpretation of probability values as relative frequencies or as degrees of belief)

likelihood interpretation

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the degree of membership $\mu_F(x)$ of x in the fuzzy set F is interpreted as (proportional to) the probability that an object with associated value x is considered as an element of F (e.g.: $\mu_{tall}(180)$ is the probability that a 180 cm person is considered as "tall", and $\mu_{tall}(195)$ is the probability that a 195 cm person is considered as "tall")

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besides probability distributions and sets, the only other descriptions of uncertain knowledge that are widely used in statistics are likelihood functions

fuzzy data

with the likelihood interpretation of membership functions, the precise observation of $Y = f(X, \varepsilon)$ (where $\varepsilon \sim F$ is an observation error) corresponds to a **fuzzy observation** of X (e.g.: X is the height of a person, and Y is the classification of the person as "tall" or not)

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fuzzy data (or fuzzified data) can lead to more robust statistical inferences

references

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