On using Different Distance Measures for Fuzzy Numbers in Fuzzy Linear Regression Models

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Outline

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then the first triangular fuzzy numbers is $\widetilde{V}_{0k} = (x_{3k}/x_{1k}/x_{2k})$.

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- Input and output data are both fuzzy (Third Category)

Fuzzy linear regression model (Second Category)

$$\widetilde{Y}_l = \widetilde{A}_0 + \widetilde{A}_1 x_{1l} + \widetilde{A}_2 x_{2l} + ... + \widetilde{A}_m x_{ml} \quad l = 1, 2, .., n \tag{1}$$

Fuzzy linear regression model (Third Category)

$$\widetilde{Y}_l = a_0 + a_1 \widetilde{X}_{1l} + a_2 \widetilde{X}_{2l} + ... + a_m \widetilde{X}_{ml} \quad l = 1, 2, .., n \tag{2}$$

Predicted values

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$$\widetilde{Y}^*_{lk} = \widetilde{V}_{0k} + \widetilde{V}_{1k}x_{1l} + \widetilde{V}_{2k}x_{2l} + ... + \widetilde{V}_{mk}x_{ml} \quad l = 1, 2, .., n \tag{3}$$

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Fuzzy linear regression model (Third Category)

$$\widetilde{Y}^*_{lk} = v_{0k} + v_{1k}\,\widetilde{X}_{1l} + v_{2k}\widetilde{X}_{2l} + ... + v_{mk}\widetilde{X}_{ml}; \quad l = 1, 2, .., n \tag{4}$$

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$$D = \int |\mu_{\tilde{Y}}(x) - \mu_{\tilde{Y}^*_{lk}}(x)| dx$$

$$E = \frac{\int_{S_{\tilde{Y}} \cup S_{\tilde{Y}^*_{lk}}} |\mu_{\tilde{Y}(x)} - \mu_{\tilde{Y}^*_{lk}(x)}| dx}{\int_{S_{\tilde{Y}}} \mu_{\tilde{Y}}(x) dx}$$

Error Measure (Abdalla & Buckley (2007))

$$E_1 = \frac{\sum_{l=1}^{n}\left[\int_{-\infty}^{\infty}|\widetilde{Y}_l(x) - \widetilde{Y}^*_{lk}(x)|dx\right]}{\left[\int_{-\infty}^{\infty}\widetilde{Y}_l(x)dx\right]} \tag{5}$$

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$$\widetilde{V}_k \in \{\widetilde{V}_1, ..., \widetilde{V}_N\} \text{ and } v_k \in \{v_1, ..., v_N\}$$

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- $\left[A^L(\alpha), A^U(\alpha)\right]$ and $\left[B^L(\alpha), B^U(\alpha)\right]$ are the closed intervals of α-cuts

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- $E_*(\widetilde{A}) = a_2 - (a_2 - a_1) \int_0^\infty L(x)dx$
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$$d_p\left(\widetilde{A}(\alpha), \widetilde{B}(\alpha)\right) =$$

$$\begin{cases} (0.5)(|A^L(\alpha) - B^L(\alpha)|^p + |A^U(\alpha) - B^U(\alpha)|^p)^{1/p}, & 1 \leq p \leq \infty; \\ max|A^L(\alpha) - B^L(\alpha)|, |A^U(\alpha) - B^U(\alpha)|, & p = \infty. \end{cases} \tag{8}$$

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$$\delta_p(\widetilde{A}, \widetilde{B}) = \begin{cases} 0.25 \left(\sum_{i=1}^{4} |a_i - b_i|^p \right)^{1/p}, & 1 \leq p < \infty; \\ max(|a_i - b_i|), & p = \infty. \end{cases} \tag{9}$$

Chen & Hsieh (1998)

- $P(A) = \dfrac{\int_0^w \alpha \left(\frac{L^{-1}(\alpha)+R^{-1}(\alpha)}{2} \right) d\alpha}{\int_0^w \alpha d\alpha}$

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Table: Data for the application (Second category)

Fuzzy Output	x_{1l}	x_{2l}	x_{3l}
(2.27/5.83/9.39)	2.00	0.00	15.25
(0.33/0.85/1.37)	0.00	5.00	14.13
(5.43/13.93/22.43)	1.13	1.50	14.13
(1.56/4.00/6.44)	2.00	1.25	13.63
(0.64/1.65/2.66)	2.19	3.75	14.75
(0.62/1.58/2.54)	0.25	3.50	13.75
(3.19/8.18/13.17)	0.75	5.25	15.25
(0.72/1.85/2.98)	4.25	2.00	13.50

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Table: Intervals for $I_i, i = 0, 1, 2, 3$ for second category

Interval	*MCI*	*MCII*	*MCIII*	*MCIV*
I_0	[-1,0]	[0,1]	[-18.174,-18.174]	[28.000,47.916]
I_1	[-1,0]	[-1,0]	[-1.083,-1.083]	[-2.542,-2.542]
I_2	[-1.5,-0.5]	[-1.5,-0.5]	[-1.150,-1.150]	[-2.323,-2.323]
I_3	[0,1]	[0,1]	[1.733,2.149]	[-1.354,-1.354]

Results for using different definitions of distance measures in fuzzy linear regression with MC method for minimizing E_1

Definitions	Parameters	Intervals											
		MCI			MCII			MCIII			MCIV		
Kaufmann *(1991)*	$\tilde{A}_0$	-0.8530	-0.5900	-0.2935	0.0607	0.3163	0.3414	-18.1740	-18.1740	-18.1740	31.0713	31.5636	32.1763
	$\tilde{A}_1$	-0.6934	-0.6033	-0.3096	-0.2712	-0.2684	-0.1293	-1.0830	-1.0830	-1.0830	-2.5420	-2.5420	-2.5420
	$\tilde{A}_2$	-1.4064	-1.3966	-1.3162	-0.8220	-0.7265	-0.7210	-1.1500	-1.1500	-1.1500	-2.3230	-2.3230	-2.3230
	$\tilde{A}_3$	0.5474	0.5727	0.5923	0.2591	0.2938	0.3359	1.7337	1.7519	1.8307	-1.3540	-1.3540	-1.3540
Heilpern-1 (1997)	$\tilde{A}_0$	-0.8472	-0.7690	-0.1782	0.0653	0.3254	0.3424	-18.1740	-18.1740	-18.1740	28.6932	30.4576	35.6408
	$\tilde{A}_1$	-0.8527	-0.3606	-0.0810	-0.8627	-0.4147	-0.0858	-1.0830	-1.0830	-1.0830	-2.5420	-2.5420	-2.5420
	$\tilde{A}_2$	-1.4198	-1.1616	-0.5778	-1.4075	-1.2370	-0.6181	-1.1500	-1.1500	-1.1500	-2.3230	-2.3230	-2.3230
	$\tilde{A}_3$	0.0251	0.6431	0.7575	0.1463	0.4066	0.7275	1.7339	1.7583	1.7678	-1.3540	-1.3540	-1.3540
Heilpern-2 (1997)	$\tilde{A}_0$	-0.8530	-0.5900	-0.2935	0.0607	0.3163	0.3414	-18.1740	-18.1740	-18.1740	31.0713	31.5636	32.1763
	$\tilde{A}_1$	-0.6934	-0.6033	-0.3096	-0.2712	-0.2684	-0.1293	-1.0830	-1.0830	-1.0830	-2.5420	-2.5420	-2.5420
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	$\tilde{A}_3$	0.5474	0.5727	0.5923	0.2591	0.2938	0.3359	1.7337	1.7519	1.8307	-1.3540	-1.3540	-1.3540
Heilpern-3 (1997)	$\tilde{A}_0$	-0.8530	-0.5900	-0.2935	0.0607	0.3163	0.3414	-18.1740	-18.1740	-18.1740	31.0713	31.5636	32.1763
	$\tilde{A}_1$	-0.6934	-0.6033	-0.3096	-0.2712	-0.2684	-0.1293	-1.0830	-1.0830	-1.0830	-2.5420	-2.5420	-2.5420
	$\tilde{A}_2$	-1.4064	-1.3966	-1.3162	-0.8220	-0.7265	-0.7210	-1.1500	-1.1500	-1.1500	-2.3230	-2.3230	-2.3230
	$\tilde{A}_3$	0.5474	0.5727	0.5923	0.2591	0.2938	0.3359	1.7334	1.7552	1.8369	-1.3540	-1.3540	-1.3540
Chen and Hsieh (1998)	$\tilde{A}_0$	-0.7617	-0.7454	-0.5821	0.0716	0.4464	0.5536	-18.1740	-18.1740	-18.1740	28.9831	31.8476	33.2103
	$\tilde{A}_1$	-0.6857	-0.4063	-0.3824	-0.9107	-0.4521	-0.0816	-1.0830	-1.0830	-1.0830	-2.5420	-2.5420	-2.5420
	$\tilde{A}_2$	-1.3294	-1.1576	-0.5469	-1.3458	-1.1448	-0.6135	-1.1500	-1.1500	-1.1500	-2.3230	-2.3230	-2.3230
	$\tilde{A}_3$	0.2521	0.4794	0.8036	0.2596	0.3323	0.9166	1.7443	1.7445	1.7981	-1.3540	-1.3540	-1.3540

Table: Data for the application (Third category)

Fuzzy output	X_{1l}	X_{2l}
(55.4/61.6/64.7)	(5.7/6.0/6.9)	(5.4/6.3/7.1)
(50.5/53.2/58.5)	(4.0/4.4/5.1)	(4.7/5.5/5.8)
(55.7/65.5/75.3)	(8.6/9.1/9.8)	(3.4/3.6/4.0)
(61.7/64.9/74.7)	(6.9/8.1/9.3)	(5.0/5.8/6.7)
(69.1/71.7/80.0)	(8.7/9.4/11.2)	(6.5/6.8/7.1)
(49.6/52.2/57.4)	(4.6/4.8/5.5)	(6.7/7.9/8.7)
(47.7/50.2/55.2)	(7.2/7.6/8.7)	(4.0/4.2/4.8)
(41.8/44.0/48.4)	(4.2/4.4/4.8)	(5.4/6.0/6.3)
(45.7/53.8/61.9)	(8.2/9.1/10.0)	(2.7/2.8/3.2)
(45.4/53.5/58.9)	(6.0/6.7/7.4)	(5.7/6.7/7.7)

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Table: Intervals for $I_i, i = 0, 1, 2$ for third category

Interval	*MCI*	*MCII*	*MCIII*	*MCIV*
I_0	[0,5]	[0,37]	[16.528,16.528]	[33.808,36.601]
I_1	[0,6]	[0,6]	[3.558,3.982]	[1.294,3.756]
I_2	[0,4]	[0,6]	[2.575,2.575]	[0.423,0.473]

Table: Results for using different definitions of distance measures in fuzzy linear regression with MC method for minimizing E_1.

		Intervals			
Distance Measures	Parameters	*MCI*	*MCII*	*MCIII*	*MCIV*
Kaufmann (1991)	a_0	1.9138	1.8114	16.5280	33.8108
	a_1	4.7655	4.7820	3.5733	3.1333
	a_2	3.6687	3.6775	2.5750	0.4730
Heilpern-1 (1997)	a_0	2.4841	0.3650	16.5280	33.8106
	a_1	4.9058	4.8024	3.5580	2.7181
	a_2	3.4424	3.9099	2.5750	0.7430
Heilpern-2 (1997)	a_0	1.9138	1.8114	16.5280	33.8108
	a_1	4.7655	4.7820	3.5733	3.1333
	a_2	3.6687	3.6775	2.5750	0.4730
Heilpern-3 (1997)	a_0	4.4812	5.5354	16.5280	33.8111
	a_1	4.5835	4.5590	3.5580	3.0608
	a_2	3.4776	3.3425	2.5750	0.4730
Chen and Hsieh (1998)	a_0	2.1047	0.5538	16.5280	33.8086
	a_1	5.0605	5.0276	3.5580	3.0994
	a_2	3.3305	3.6148	2.5750	0.4730

Table: Error measures for application (second category)

E_1	MCI	$MCII$	$MCIII$	$MCIV$
Abdalla and Buckley (2008)	6.169	5.812	7.125	8.201
Kaufmann (1991)	32.63132	31.0182	24.1279	110.6466
Heilpern-1 (1997)	4.5126	6.8999	12.202	50.9251
Heilpern-2 (1997)	16.31566	15.5091	12.06395	55.3233
Heilpern-3 (1997)	16.3649	15.104	9.2622	40.2581
Chen and Hsieh (1998)	6.1242	4.8169	11.7306	58.7061

Table: Error measures for application (third category)

E_1	*MCI*	*MCII*	*MCIII*	*MCIV*
Abdalla and Buckley (2008)	10.017	9.389	12.7267	9.5933
Kaufmann (1991)	52.7943	83.9582	19.0558	24.3161
Heilpern-1 (1997)	26.2680	42.0170	9.4604	13.4241
Heilpern-2 (1997)	26.3971	41.9791	9.5279	12.1581
Heilpern-3 (1997)	19.8377	31.5128	7.2577	9.4778
Chen and Hsieh (1998)	26.3563	41.9412	9.4544	11.6395

Outline

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Reason

- Only one definition of distance measure has been used in fuzzy regression with Monte Carlo method until now.
- Hence, we investigate using different definitions of distance measure between fuzzy numbers in estimating the parameters of fuzzy regression with Monte Carlo method.

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