M-estimation when data values are not completely known

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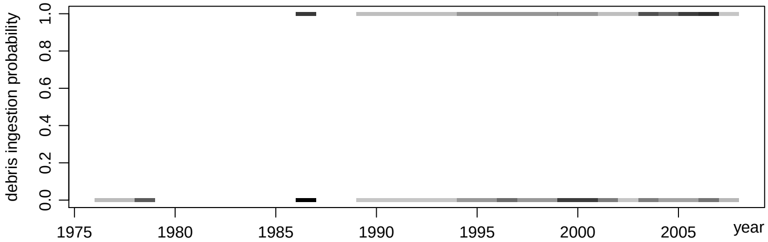
example: logistic regression with interval-censored covariate



- for 468 green turtles, the data describe the presence or absence of marine debris in the gastrointestinal system at the time of death, which is interval-censored (Schuyler et al., 2014)

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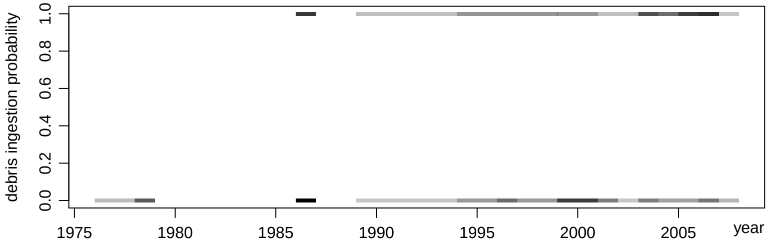

- for 468 green turtles, the data describe the presence or absence of marine debris in the gastrointestinal system at the time of death, which is interval-censored (Schuyler et al., 2014)
- the main question is if the probability of debris ingestion increased over time: we decide to use logistic regression to answer it

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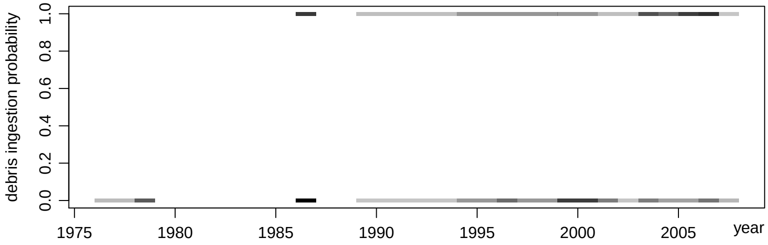

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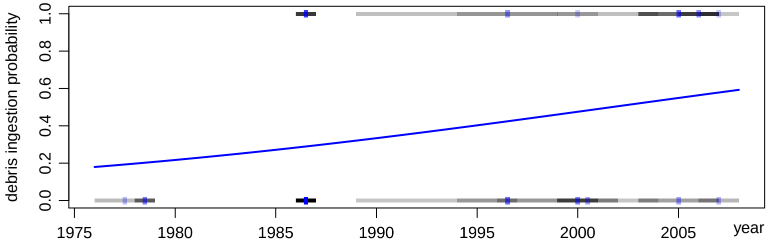

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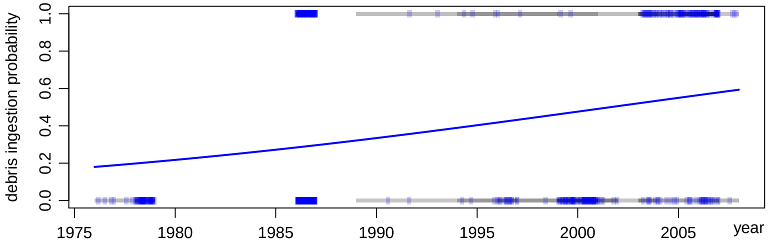

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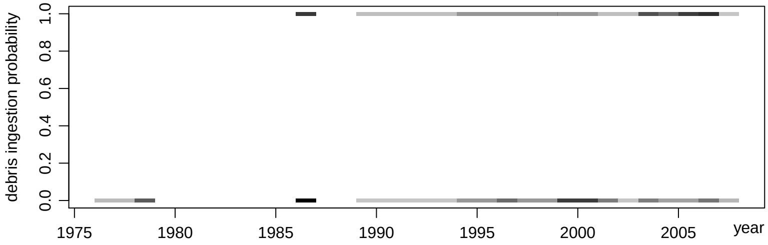

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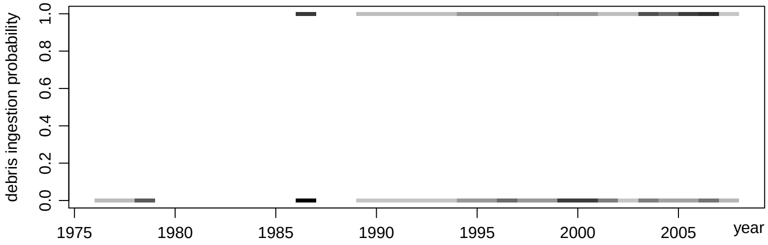

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  - apply the conventional statistical method to all compatible precise data sets: why? and how?
  - adapt the conventional statistical method to the case of incomplete/censored data: we will follow this approach

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- nonparametric/pragmatic approach: under weak regularity conditions (Huber and Ronchetti, 2009),

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- parametric/idealistic approach: assuming further $X_i \sim P_\theta$ and $E[\rho(X_i, \theta)] < E[\rho(X_i, \theta')]$,

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- parametric/idealistic approach: under additional assumptions,

$$\hat{\theta}_{\text{minimin}}(S_1, \ldots, S_n) \xrightarrow[n \to \infty]{\text{a.s.}} [\theta],$$

  where $[\theta]$ is the identification region of θ (Manski, 2003): smoothing corrections (of the ε-minimin form) may be needed in case of partial identification

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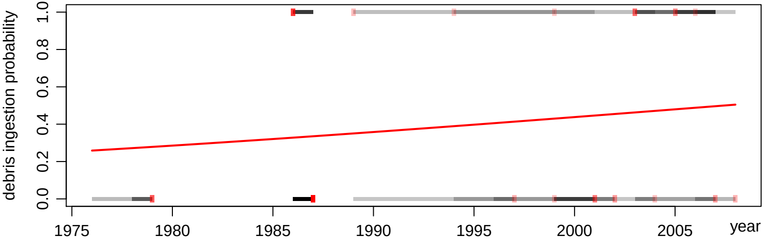

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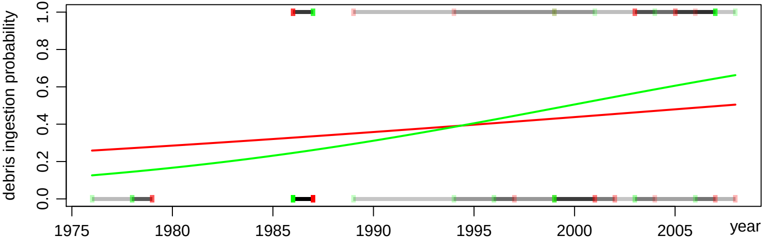

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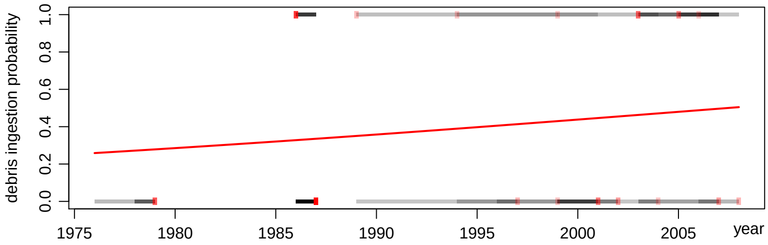

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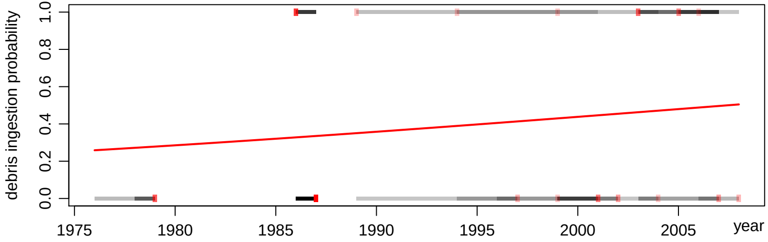

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- note however that this reasoning is not valid in general for other tests (e.g., the Wald test)

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- by contrast, the parametric/idealistic approach of finding the hypothetical true model can be pursued by minimin M-estimation, but is often severely hindered by partial identification

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