On the implementation of Likelihood-based Imprecise Regression

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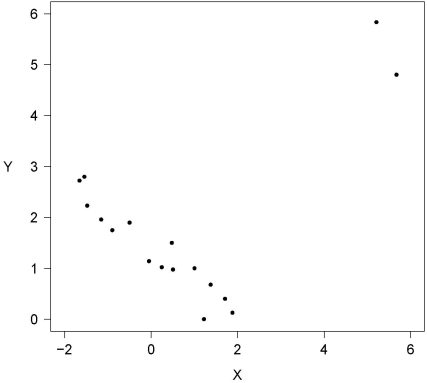
simple linear regression



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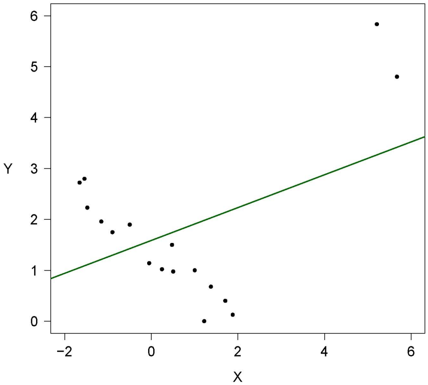

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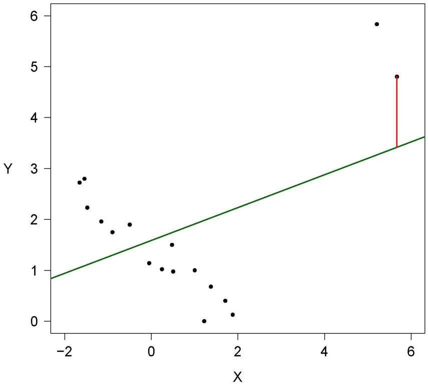

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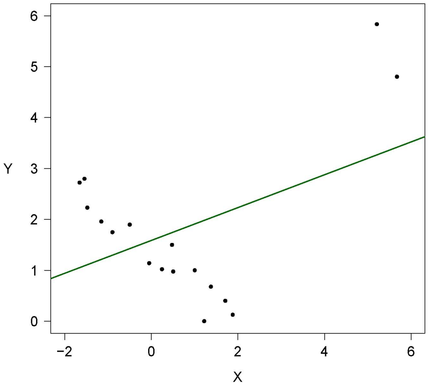

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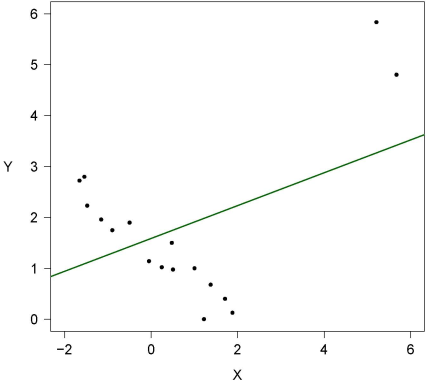

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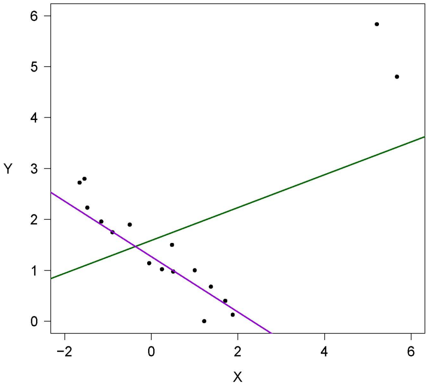

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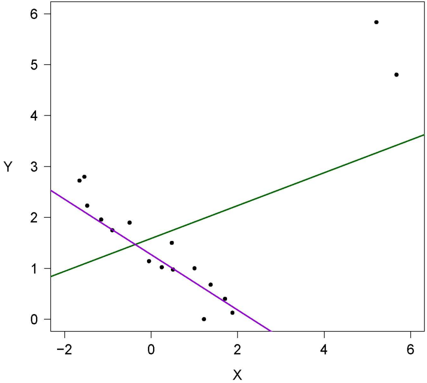

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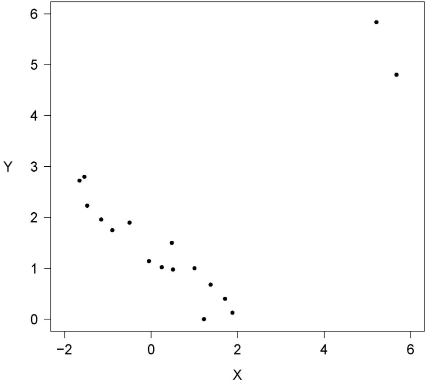
imprecisely observed data



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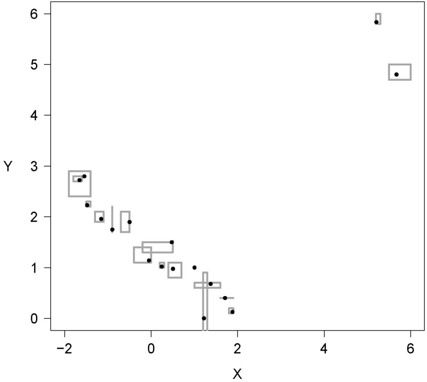

- imprecise data: $(\underline{x}_i, \overline{x}_i, \underline{y}_i, \overline{y}_i) \in \overline{\mathbb{R}}^4$ for each $i \in \{1, \ldots, n\}$

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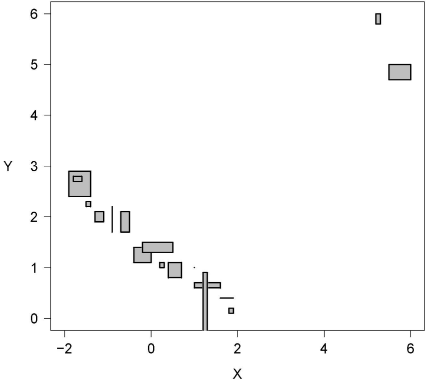

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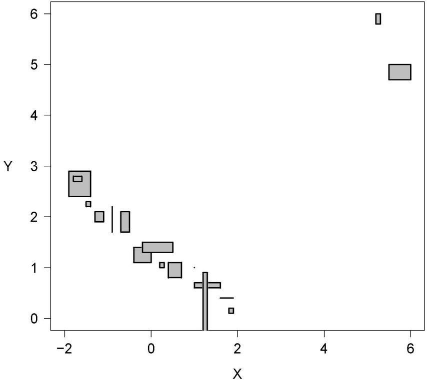

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- nonparametric statistical model: $\mathcal{P}$ is the set of all probability measures P such that $\underline{X}_i, X_i, \overline{X}_i, \underline{Y}_i, Y_i, \overline{Y}_i$ have a joint distribution satisfying

$$\underline{X}_i \leq X_i \leq \overline{X}_i \quad \text{and} \quad \underline{Y}_i \leq Y_i \leq \overline{Y}_i \quad P\text{-a.s.}$$

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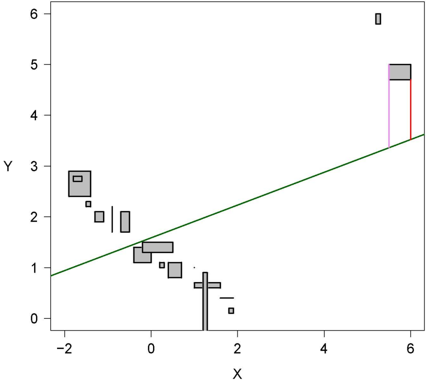

imprecise residuals:

$$\underline{r}_{f,i} = \min_{(x,y)\in[\underline{x}_i,\overline{x}_i]\times[\underline{y}_i,\overline{y}_i]} |y - f(x)|$$

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for each $f \in \mathcal{F}$, $i \in \{1, \dots, n\}$

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- likelihood function: $lik : P \mapsto \prod_{i=1}^{n} \dfrac{P(\underline{X}_i = \underline{x}_i,\, \overline{X}_i = \overline{x}_i,\, \underline{Y}_i = \underline{y}_i,\, \overline{Y}_i = \overline{y}_i)}{\widehat{P}_{\underline{X},\overline{X},\underline{Y},\overline{Y}}(\underline{x}_i, \overline{x}_i, \underline{y}_i, \overline{y}_i)}$

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$$\underline{\mathit{med}} R_f = \underline{r}_{f,(\underline{k}+1)} \quad \text{and} \quad \overline{\mathit{med}} R_f = \overline{r}_{f,(\overline{k})},$$

where $\sqrt[n]{\beta} \mapsto \frac{\overline{k}}{n}$ is a decreasing bijection $[\frac{1}{2}, 1) \to (\frac{1}{2}, 1]$, and $\underline{k} = n - \overline{k}$

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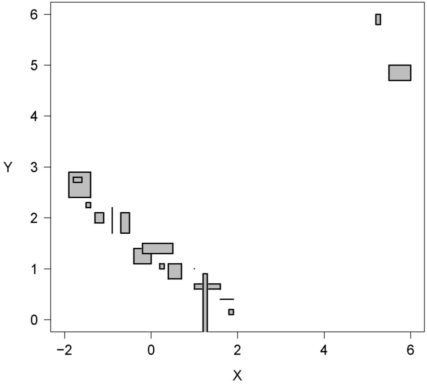
- interval dominance: $\mathcal{U} = \{f \in \mathcal{F} : \underline{med}\, R_f \leq \overline{med}\, R_{f_{LRM}}\}$ is the set of all undominated regression lines

algorithm for f_{LRM}



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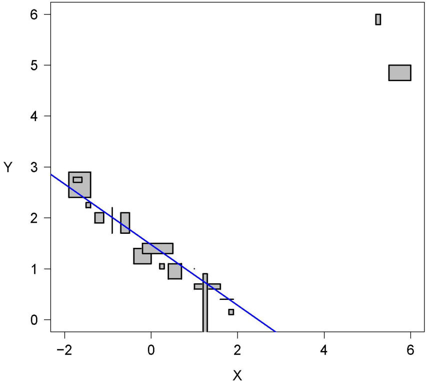

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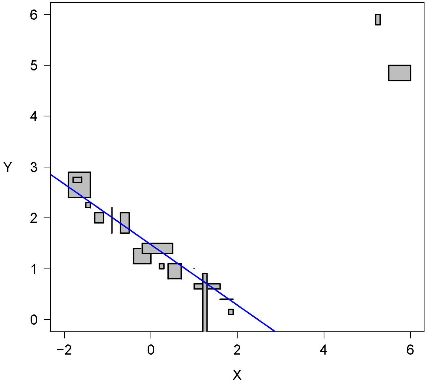

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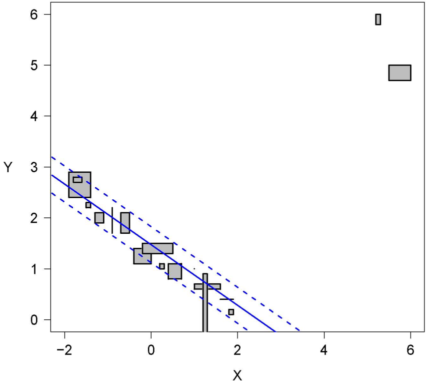

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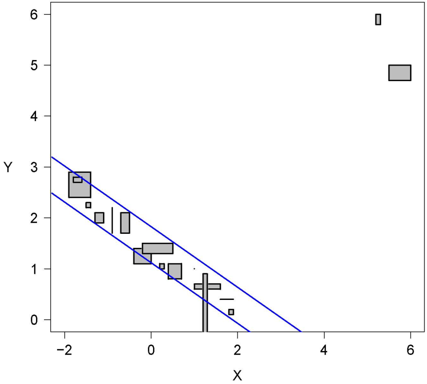

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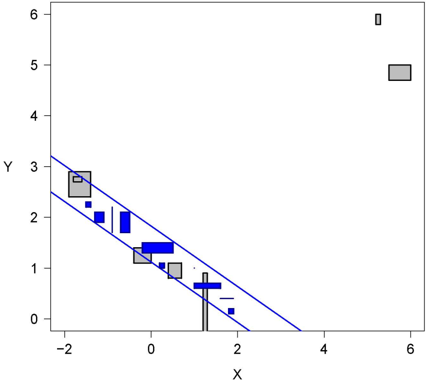

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- $f_{LRM} \pm \overline{med}\, R_{f_{LRM}}$ is the thinnest strip of the form $f \pm q$ containing (at least) $\overline{k}$ imprecise data $[\underline{x}_i, \overline{x}_i] \times [\underline{y}_i, \overline{y}_i]$, for all $f \in \mathcal{F}$, $q \in [0, +\infty)$

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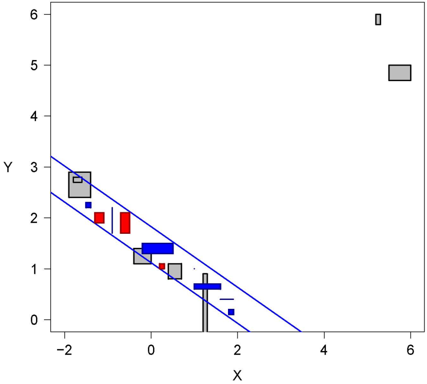

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- if the slope $b_{LRM} \neq 0$, then the imprecise data contained in $f_{LRM} \pm \overline{med} R_{f_{LRM}}$ are bounded and (at least) 3 of them touch the boundary of the strip

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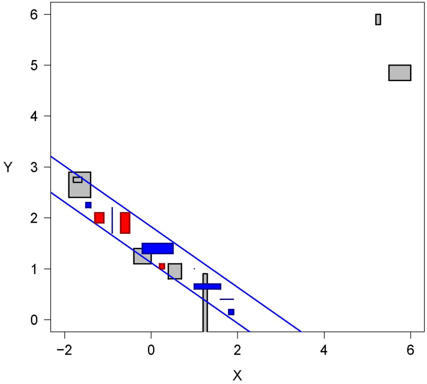

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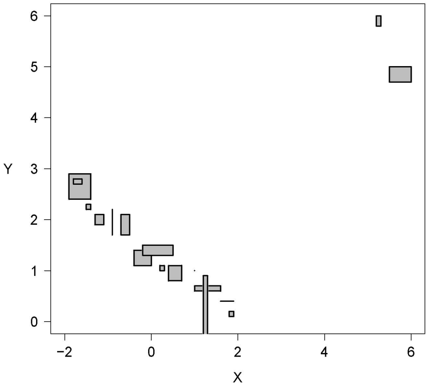
- if the slope $b_{LRM} \neq 0$, then the imprecise data contained in $f_{LRM} \pm \overline{med} R_{f_{LRM}}$ are bounded and (at least) 3 of them touch the boundary of the strip
- therefore, b_{LRM} is either 0 or it is determined by a couple of bounded imprecise data, which gives us at most $4\binom{n}{2} + 1$ possible values for b_{LRM}

undominated regression lines



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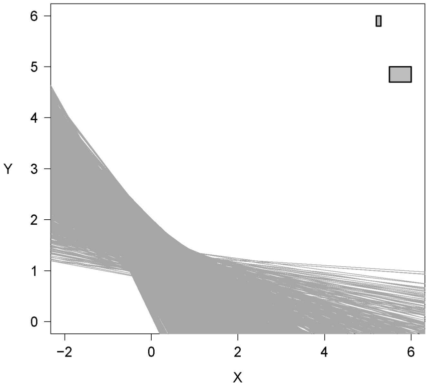

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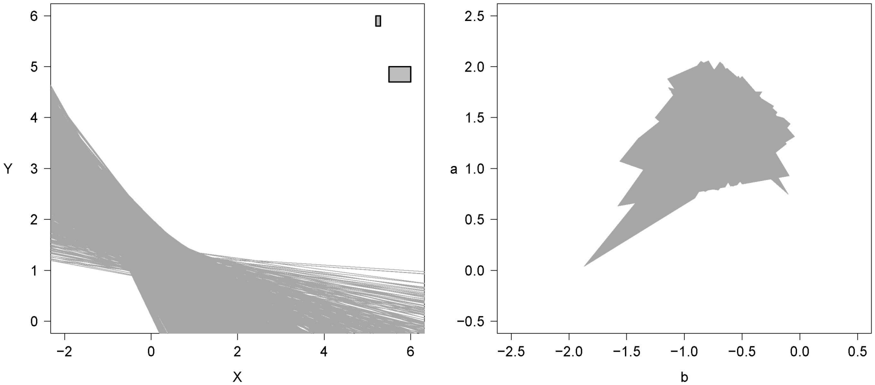
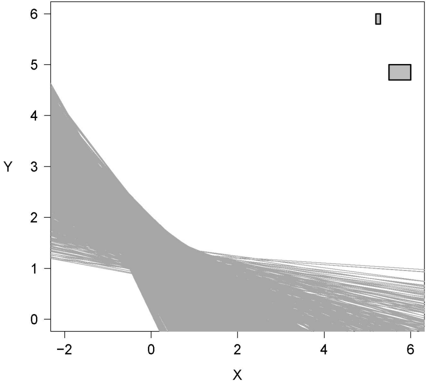
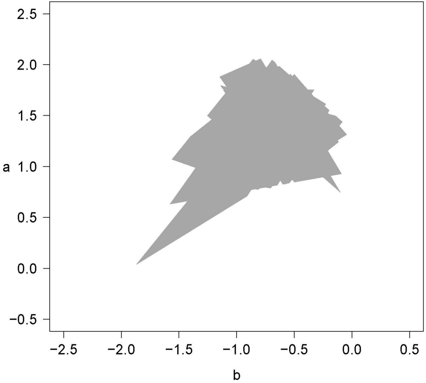


- set of undominated parameters: $\left\{(a, b) \in \mathbb{R}^2 : f_{a,b} \in \mathcal{U}\right\}$

undominated regression lines





- set of undominated parameters: $\left\{(a,b) \in \mathbb{R}^2 : f_{a,b} \in \mathcal{U}\right\} =$

$$= \bigcup_{i=1}^{\overline{k}} \left\{(a,b) \in \mathbb{R}^2 : \underline{d}_{b,(i+\underline{k})} - \overline{med} R_{f_{LRM}} \leq a \leq \overline{d}_{b,(i)} + \overline{med} R_{f_{LRM}} \right\},$$

where $\underline{d}_{b,i} = \inf_{x \in [\underline{x}_i, \overline{x}_i]} (\underline{y}_i - b\,x)$ and $\overline{d}_{b,i} = \sup_{x \in [\underline{x}_i, \overline{x}_i]} (\overline{y}_i - b\,x)$

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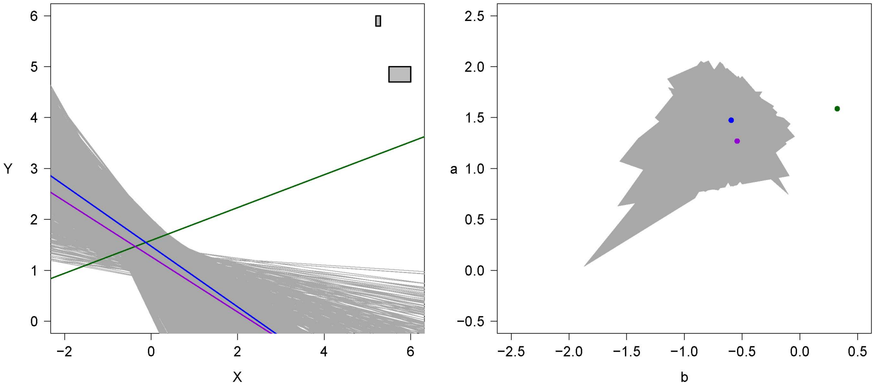

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- for example: $\overline{med}\,R_{f_{LRM}} \approx 0.354$, $\underline{med}\,R_{f_{LMS}} \approx 0.002$, $\underline{med}\,R_{f_{LS}} \approx 0.909$

statistical properties of LIR

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β	n	$\underline{P}(\underline{med}\, R_f \leq med\, R_f \leq \overline{med}\, R_f)$	F_0	$\underline{P}(f_{a_0,b_0} \in \mathcal{U})$
0.5	20	0.737	*Normal*	0.83
			Cauchy	0.97
	1000	0.758	*Normal*	1.00
			Cauchy	1.00
0.75	20	0.497	*Normal*	0.39
			Cauchy	0.72
	1000	0.533	*Normal*	0.91
			Cauchy	1.00
0.999	20	0.176	*Normal*	0.03
			Cauchy	0.11
	1000	0.025	*Normal*	0.00
			Cauchy	0.01

references

Cattaneo, M., and Wiencierz, A. (2011*a*). Likelihood-based Imprecise Regression. Technical Report 116. Department of Statistics, LMU Munich.

Cattaneo, M., and Wiencierz, A. (2011*b*). Regression with Imprecise Data: A Robust Approach. In *ISIPTA '11, Proceedings of the Seventh International Symposium on Imprecise Probability: Theories and Applications*. SIPTA, 119–128.

Cattaneo, M., and Wiencierz, A. (2011*c*). Robust regression with imprecise data. Technical Report 114. Department of Statistics, LMU Munich.

Hampel, F. R., Ronchetti, E. M., Rousseeuw, P. J., and Stahel, W. A. (1986). *Robust Statistics: The Approach Based on Influence Functions*. Wiley.

Rousseeuw, P. J., and Leroy, A. M. (1987). *Robust Regression and Outlier Detection*. Wiley.

Steele, J., and Steiger, W. (1986). Algorithms and complexity for least median of squares regression. *Discrete Appl. Math.* 14, 93–100.