The likelihood approach to statistics

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past and present research

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  (reasoning and decision making under uncertainty)

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- Postdoc with Thomas Augustin at LMU Munich
(SNSF Research Fellowship, October 2007 – March 2009):
Decision making on the basis of a probabilistic-possibilistic hierarchical description of uncertain knowledge

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- the likelihood function is a central concept in statistics (both frequentist and Bayesian)

statistical inference

- the likelihood approach allows conditional inference without needing precise prior information

$$\begin{tikzcd}
\theta \arrow[d] \arrow[dr] \arrow[drrr] & & & \\
X_1 & X_2 & \cdots & X_n
\end{tikzcd}$$

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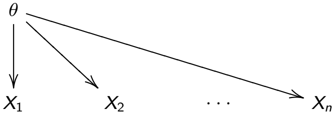
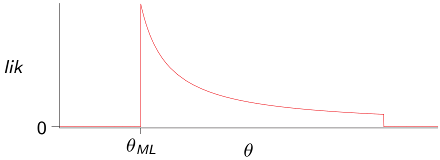


- likelihood-based inference methods are very successful, in particular **maximum likelihood** estimates (parametric or nonparametric)



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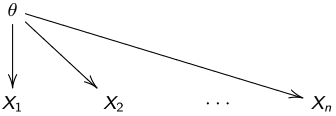

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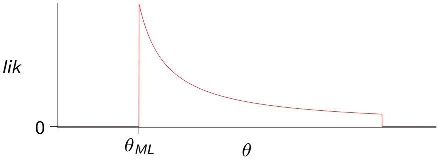

- surprisingly, no likelihood-based decision making in the statistical literature

statistical decisions

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$$L : \Theta \times \mathcal{D} \to [0, \infty],$$

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- likelihood-based decision criterion:

$$\text{minimize} \quad \int L(\theta, d) \,\mathrm{d}\lambda(\theta)$$

likelihood-based statistical decisions

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- a simple property (sure-thing principle) characterizes the likelihood-based decision criterion based on the Shilkret integral: the **MPL** (Minimax Plausibility-weighted Loss) decision criterion:

$$\text{minimize} \quad \int^{\mathrm{S}} L(\theta, d)\, \mathrm{d}\lambda(\theta) = \sup_{\theta \in \Theta} lik(\theta)\, L(\theta, d)$$

example: estimation of variance components

- estimation of the variance components in the 3×3 random effect one-way layout, under normality assumptions and weighted squared error loss

$$X_{ij} = \mu + \alpha_i + \varepsilon_{ij} \quad \text{for all } i, j \in \{1, 2, 3\}$$

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- normality assumptions:

$$\alpha_i \sim \mathcal{N}(0, v_a), \;\; \varepsilon_{ij} \sim \mathcal{N}(0, v_e), \;\; \text{all independent}$$

$$\Rightarrow \;\; X_{ij} \sim \mathcal{N}(\mu, v_a + v_e) \;\text{dependent}, \;\; \mu \in (-\infty, \infty), \;\; v_a, v_e \in (0, \infty)$$

example: estimation of variance components

- estimates $\widehat{v_e}$ and $\widehat{v_a}$ of variance components v_e and v_a are functions of

$$SS_e = \sum_{i=1}^{3} \sum_{j=1}^{3} (x_{ij} - \bar{x}_{i\cdot})^2 \quad \text{and} \quad SS_a = 3 \sum_{i=1}^{3} (\bar{x}_{i\cdot} - \bar{x}_{\cdot\cdot})^2,$$

where

$$\bar{x}_{i\cdot} = \frac{1}{3} \sum_{j=1}^{3} x_{ij}, \quad \bar{x}_{\cdot\cdot} = \frac{1}{9} \sum_{i=1}^{3} \sum_{j=1}^{3} x_{ij},$$

$$\frac{SS_e}{v_e} \sim \chi^2_6, \quad \text{and} \quad \frac{\frac{1}{3}\, SS_a}{v_a + \frac{1}{3}\, v_e} \sim \chi^2_2$$

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- invariant loss functions:

$$L(v_e, \widehat{v_e}) = 3\,\frac{(v_e - \widehat{v_e})^2}{{v_e}^2} \quad \text{and} \quad L(v_a, \widehat{v_a}) = \frac{(v_a - \widehat{v_a})^2}{(v_a + \frac{1}{3}\,v_e)^2}$$

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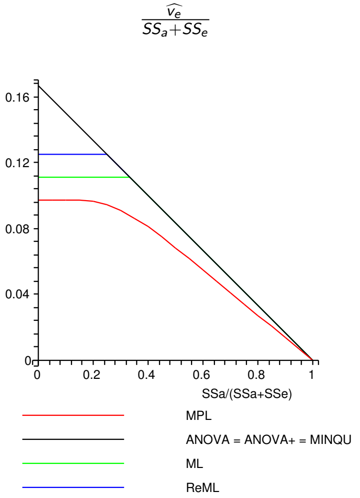

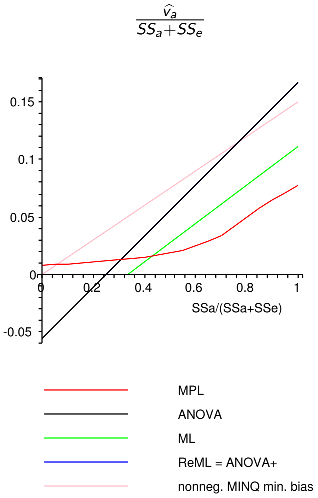

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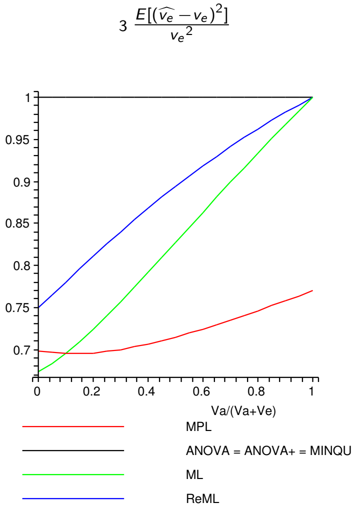

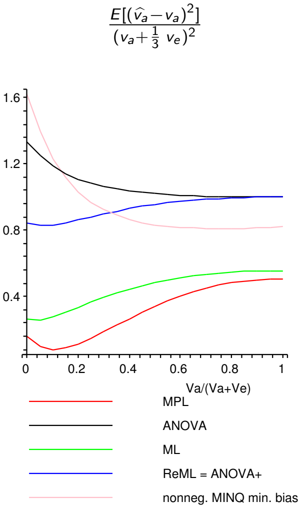

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  (probabilistic and non-probabilistic aspects of uncertainty)
- application to financial risk measures
  (derivation and interpretation of convex risk measures)