# On the validity of minimin and minimax methods for Support Vector Regression with interval data

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## Support Vector Regression (SVR) with precise data

**data**: $(x_1, y_1), \dots, (x_n, y_n) \in \mathcal{X} \times \mathcal{Y} \overset{\text{compact}}{\subset} \mathbb{R}^d \times \mathbb{R}$

**Reproducing Kernel Hilbert Space**: set $\mathcal{F}$ of functions $f : \mathcal{X} \to \mathcal{Y}$, e.g., with the Gaussian kernel $\kappa_\sigma$ defined for all $x, x' \in \mathcal{X}$ and $\sigma > 0$ by

$$\kappa_\sigma(x, x') = \exp\left(-\tfrac{1}{\sigma^2}\|x - x'\|^2\right),$$

$\mathcal{F}$ is dense in the space $\mathcal{C}(\mathcal{X})$ of continuous functions

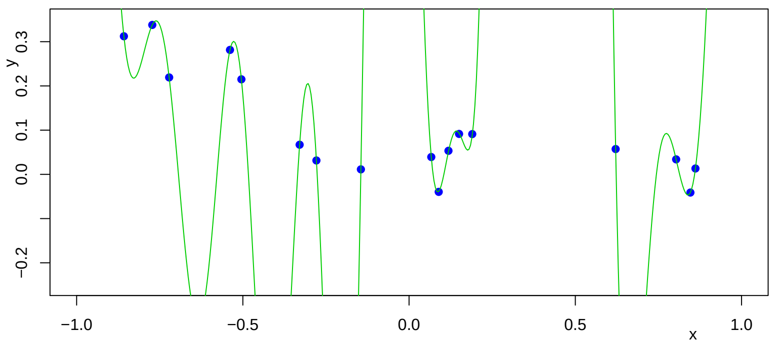
**regression function**: find the function $f \in \mathcal{F}$ that best describes the relationship between the variables of interest in the light of the data

**general idea**: function $f \in \mathcal{F}$ minimizing the (empirical) risk

$$\mathcal{E}(f) = \frac{1}{n} \sum_{i=1}^{n} \psi\left(|y_i - f(x_i)|\right),$$

where $\psi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is convex with $\psi(0) = 0$, e.g., for the linear loss $\psi$ is defined by $\psi(r) = r$ for all $r \in \mathbb{R}_{\geq 0}$

$\rightsquigarrow$ estimated functions are too wiggly when considering large $\mathcal{F}$

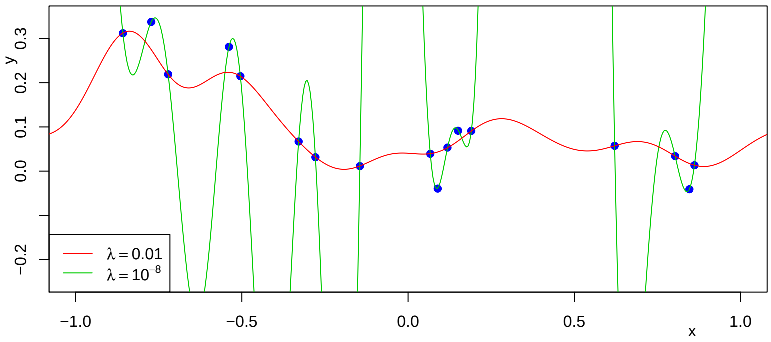



Unpenalized regression function based on Gaussian kernel and linear loss with precise data $(x_i, y_i) \in \mathbb{R}^2$ where $i \in \{1, \dots, 17\}$

**SVR estimate**: function $f \in \mathcal{F}$ minimizing the regularized risk

$$\mathcal{E}_\lambda(f) = \frac{1}{n}\sum_{i=1}^{n} \psi\left(|y_i - f(x_i)|\right) + \lambda \, \|f\|_{\mathcal{F}}^2,$$

where $\psi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is convex with $\psi(0) = 0$, and $\lambda \in \mathbb{R}_{>0}$




Unpenalized regression function vs. SVR estimate, both based on Gaussian kernel and linear loss

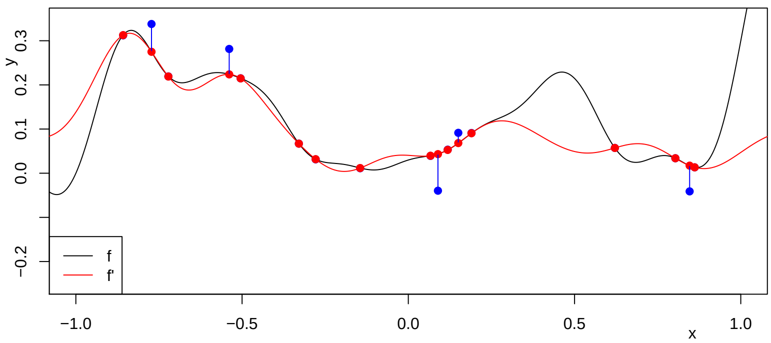
**Representer Theorem (RT)**: the regression function minimizing $\mathcal{E}_\lambda(f)$ exists, is unique, and has the form

$$f = \sum_{j=1}^{n} \alpha_j \, \kappa(\,\cdot\,, x_j),$$

where $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$, and $\kappa$ is the kernel function associated with $\mathcal{F}$

**key result underlying the SVR methodology**: the minimization of $\mathcal{E}_\lambda(f)$ becomes a convex optimization task in $n$ variables $\alpha_1, \ldots, \alpha_n$, i.e., the RT makes the theoretical idea practically feasible

**core of the proof** (see e.g., Steinwart & Christmann (2008)): the structure of $\mathcal{F}$ implies that for each $f$, the orthogonal projection $f' = \sum_{j=1}^{n} \alpha_j' \, \kappa(\,\cdot\,, x_j)$ of $f$ on the subspace spanned by the functions $\kappa(\,\cdot\,, x_j)$ satisfies $f'(x_i) = f(x_i)$ for all $i \in \{1, \ldots, n\}$, and therefore $\mathcal{E}_\lambda(f') \leq \mathcal{E}_\lambda(f)$




SVR estimate $f'$ based on Gaussian kernel and linear loss vs. another $f \in \mathcal{F}$ with $f(x_i) = f'(x_i)$ for all $i \in \{1, \ldots, 17\}$

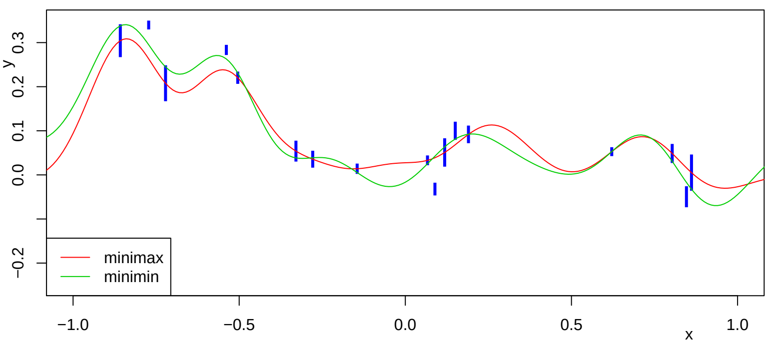
## minimin and minimax methods for SVR with interval-valued response

**interval data**: instead of the values $y_1, \dots, y_n$, only intervals $[\underline{y}_1, \overline{y}_1], \dots, [\underline{y}_n, \overline{y}_n]$ are observed, with $y_i \in [\underline{y}_i, \overline{y}_i]$ for all $i \in \{1, \dots, n\}$

**minimin and minimax SVR estimates** (Utkin & Coolen (2011)): $f \in \mathcal{F}$ minimizing

$$\underline{\mathcal{E}}_\lambda(f) = \frac{1}{n} \sum_{i=1}^{n} \min_{y_i \in [\underline{y}_i, \overline{y}_i]} \psi\left(|y_i - f(x_i)|\right) + \lambda \, \|f\|_{\mathcal{F}}^2 \quad \text{and}$$

$$\overline{\mathcal{E}}_\lambda(f) = \frac{1}{n} \sum_{i=1}^{n} \max_{y_i \in [\underline{y}_i, \overline{y}_i]} \psi\left(|y_i - f(x_i)|\right) + \lambda \, \|f\|_{\mathcal{F}}^2$$




minimin SVR estimate vs. minimax SVR estimate, both based on Gaussian kernel and linear loss

## RT for minimin and minimax SVR

**Lemma 1.** *The regularized lower and upper risk functionals, $\underline{\mathcal{E}}_\lambda$ and $\overline{\mathcal{E}}_\lambda$, respectively have unique minimizers $f_\lambda^{\text{minimin}}$ and $f_\lambda^{\text{minimax}}$ in $\mathcal{F}$, respectively.*

*Proof.* The proof can be found in the paper. □

**Theorem 1.** *There are $\alpha_1^{\text{minimin}}, \dots, \alpha_n^{\text{minimin}} \in \mathbb{R}$ and $\alpha_1^{\text{minimax}}, \dots, \alpha_n^{\text{minimax}} \in \mathbb{R}$ such that*

$$f_\lambda^{\text{minimin}} : x \mapsto \sum_{i=1}^{n} \alpha_i^{\text{minimin}} \, \kappa(x, x_i) \quad \textit{and}$$

$$f_\lambda^{\text{minimax}} : x \mapsto \sum_{i=1}^{n} \alpha_i^{\text{minimax}} \, \kappa(x, x_i)$$

*are the unique minimizers of $\underline{\mathcal{E}}_\lambda$ and $\overline{\mathcal{E}}_\lambda$ in $\mathcal{F}$, respectively.*

*Proof.* Let $f'$ denote the orthogonal projection of a function $f \in \mathcal{F}$ on the subspace $\mathcal{F}_n$ spanned by the functions $\kappa(\cdot, x_i)$ with $i \in \{1, \dots, n\}$. Then $\|f'\|_{\mathcal{F}} \leq \|f\|_{\mathcal{F}}$, and $f'$ is of the form $\sum_{i=1}^{n} \alpha_i \, \kappa(\cdot, x_i)$ with $\alpha_1, \dots, \alpha_n \in \mathbb{R}$. Moreover, for each $i \in \{1, \dots, n\}$, the orthogonality of $f' - f$ and $\kappa(\cdot, x_i)$ implies $f'(x_i) = f(x_i)$, because

$$f'(x_i) - f(x_i) = \langle f' - f, \, \kappa(\cdot, x_i) \rangle_{\mathcal{F}} = 0.$$

Therefore, $\underline{\mathcal{E}}_\lambda(f') \leq \underline{\mathcal{E}}_\lambda(f)$ and $\overline{\mathcal{E}}_\lambda(f') \leq \overline{\mathcal{E}}_\lambda(f)$, and the desired result is implied by Lemma 1. □

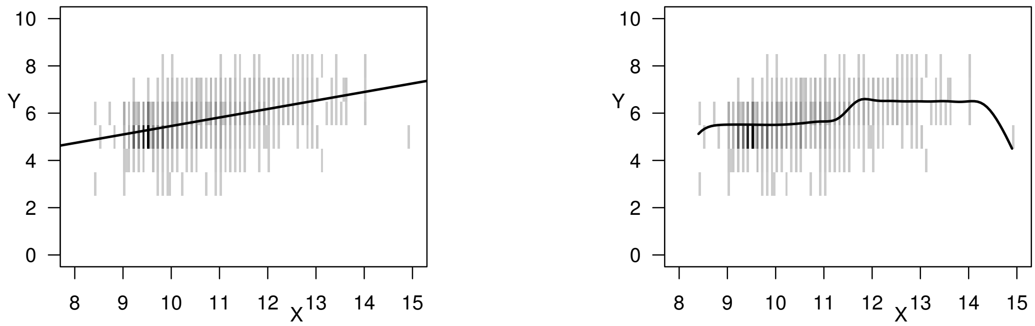
## SVR analysis of wine quality

**wine data**: we analyze the red wine sample ($n = 1\,599$) of the Vinho Verde wine data set initially analyzed by Cortez et al. (2009), which is freely available from the UC Irvine Machine Learning Repository (`http://archive.ics.uci.edu/ml/`)

**relationship of interest**: between alcohol content (explanatory variable) and sensory quality (interval-valued response) of a red wine

**results**: both minimax SVR analyses suggest an increasing relationship

- SVR with linear kernel and Least Squares (LS) loss (a.k.a. Ridge regression)
- SVR with Gaussian kernel and linear loss




minimax SVR estimates based on linear kernel and LS loss (left), i.e., $\kappa(x, x') = \langle x, x' \rangle + 1$ for all $x, x' \in \mathcal{X}$ and $\psi(r) = r^2$ for all $r \in \mathbb{R}_{\geq 0}$, and based on Gaussian kernel and linear loss (right)

## conclusions

**main contribution of the paper**: generalization of the RT to the case with interval data $[\underline{y}_1, \overline{y}_1], \dots, [\underline{y}_n, \overline{y}_n] \subset \mathbb{R}$, justifying minimin and minimax SVR in this case

**no further generalization**: the RT for interval-valued response cannot be directly generalized to the case with interval data $[\underline{x}_1, \overline{x}_1], \dots, [\underline{x}_n, \overline{x}_n] \subset \mathbb{R}^d$, in which the following expressions have to be minimized

$$\underline{\mathcal{E}}_\lambda(f) = \frac{1}{n} \sum_{i=1}^{n} \min_{x_i \in [\underline{x}_i, \overline{x}_i]} \psi\left(|y_i - f(x_i)|\right) + \lambda \, \|f\|_{\mathcal{F}}^2 \quad \text{and}$$

$$\overline{\mathcal{E}}_\lambda(f) = \frac{1}{n} \sum_{i=1}^{n} \max_{x_i \in [\underline{x}_i, \overline{x}_i]} \psi\left(|y_i - f(x_i)|\right) + \lambda \, \|f\|_{\mathcal{F}}^2$$

- a regression function minimizing $\underline{\mathcal{E}}_\lambda(f)$ would have the form $f = \sum_{j=1}^{n} \alpha_j \, \kappa(\,\cdot\,, x_j)$, where $\alpha_j \in \mathbb{R}$ and $x_j \in [\underline{x}_j, \overline{x}_j]$ for all $j \in \{1, \dots, n\}$, but in general $\underline{\mathcal{E}}_\lambda$ is not convex
- by contrast, $\overline{\mathcal{E}}_\lambda$ is convex, but a regression function minimizing $\overline{\mathcal{E}}_\lambda(f)$ does not necessarily have the form $f = \sum_{j=1}^{n} \alpha_j \, \kappa(\,\cdot\,, x_j)$, where $\alpha_j \in \mathbb{R}$ and $x_j \in [\underline{x}_j, \overline{x}_j]$ for all $j \in \{1, \dots, n\}$

**the even more general case** with interval data $[\underline{x}_i, \overline{x}_i] \times [\underline{y}_i, \overline{y}_i] \subset \mathbb{R}^d \times \mathbb{R}$ for all $i \in \{1, \dots, n\}$ also presents the above difficulties

## references

Cortez, P., Cerdeira, A., Almeida, F., Matos, T., and Reis, J. (2009). Modeling wine preferences by data mining from physicochemical properties. *Decision Support Systems* 47, 547–553.

Steinwart, I., and Christmann, A. (2008). *Support Vector Machines*. Springer.

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