A Generalization of Credal Networks

Marco Cattaneo
Department of Statistics, LMU Munich
cattaneo@stat.uni-muenchen.de

July 17, 2009

hierarchical model:

- description with **two levels**:

  probabilistic level: set $\mathcal{P}$ of probability measures P on $(\Omega, \mathcal{A})$

  possibilistic level: likelihood function $lik : \mathcal{P} \to (0\,, \infty)$

- description as **set of measures**:

  set $\mathcal{M}$ of measures μ on $(\Omega, \mathcal{A})$ with $\mu(\Omega) \in (0, \infty)$

connection between the two descriptions:

- two levels $\rightsquigarrow$ set of measures:

$$\mathcal{M} = \{lik(P)\, P : P \in \mathcal{P}\}$$

- set of measures $\rightsquigarrow$ two levels:

  probabilistic level: $\mathcal{P} = \left\{ \frac{\mu}{\mu(\Omega)} : \mu \in \mathcal{M} \right\}$

  possibilistic level: $lik : \mathcal{P} \to (0\,, \infty)$ with

$$lik(P) \propto \sup_{\mu \in \mathcal{M} \,:\, \frac{\mu}{\mu(\Omega)} = P} \mu(\Omega)$$

updating of hierarchical model (event $A \in \mathcal{A}$ observed):

- description with two levels:

probabilistic level: $\mathcal{P} \rightsquigarrow \mathcal{P}' = \{P(\cdot \mid A) : P \in \mathcal{P},\ P(A) > 0\}$

possibilistic level: $lik \rightsquigarrow lik' : \mathcal{P}' \to (0\,,\infty)$ with

$$lik'(P') \propto \sup_{P \in \mathcal{P} \,:\, P(\cdot \mid A) = P'} lik(P)\, P(A)$$

- description as set of measures:

$$\mathcal{M} \rightsquigarrow \mathcal{M}' = \{\mu(\cdot \cap A) : \mu \in \mathcal{M},\ \mu(A) > 0\}$$

in particular (**Theorem 2**): if $\mathcal{M} = \mathrm{ch}\left(\{\mu_i : i \in \{1, \ldots, m\}\}\right)$, then $\mathcal{M}' = \mathrm{ch}\left(\{\mu_i(\cdot \cap A) : i \in \{1, \ldots, m\}\,,\ \mu_i(A) > 0\}\right)$

hierarchical network:

directed acyclic graph with nodes $X_1 \in \mathcal{X}_1, \ldots, X_k \in \mathcal{X}_k$, such that to each node X_i are associated a set $\mathcal{P}_i$ of stochastic kernels P_i from $\mathcal{PA}_i$ (the set of all possible values of the parents of X_i) to $\mathcal{X}_i$, and a likelihood function $lik_i : \mathcal{P}_i \to (0\,,\infty)$

- description with two levels:

prob. level: $\mathcal{P} = \left\{P_{P_1,\ldots,P_k} : P_1 \in \mathcal{P}_1, \ldots, P_k \in \mathcal{P}_k\right\}$ with

$$P_{P_1,\ldots,P_k}(x_1, \ldots, x_k) = \prod_{i=1}^{k} P_i\left(x_i \mid pa_i(x_1, \ldots, x_k)\right)$$

poss. level: $lik : \mathcal{P} \to (0\,,\infty)$ with

$$lik(P) \propto \sup_{\substack{P_1 \in \mathcal{P}_1, \ldots, P_k \in \mathcal{P}_k \,: \\ P_{P_1,\ldots,P_k} = P}} \prod_{i=1}^{k} lik_i(P_i)$$

- description as set of measures:

$$\mathcal{M} = \left\{\mu_{P_1,\ldots,P_k} : P_1 \in \mathcal{P}_1, \ldots, P_k \in \mathcal{P}_k\right\} \text{ with}$$

$$\mu_{P_1,\ldots,P_k}(x_1, \ldots, x_k) = \prod_{i=1}^{k} \left[lik_i(P_i)\, P_i\left(x_i \mid pa_i(x_1, \ldots, x_k)\right)\right]$$

inference about the value $g(P)$ of $g : \mathcal{P} \to \mathcal{G}$ (**extension principle**):

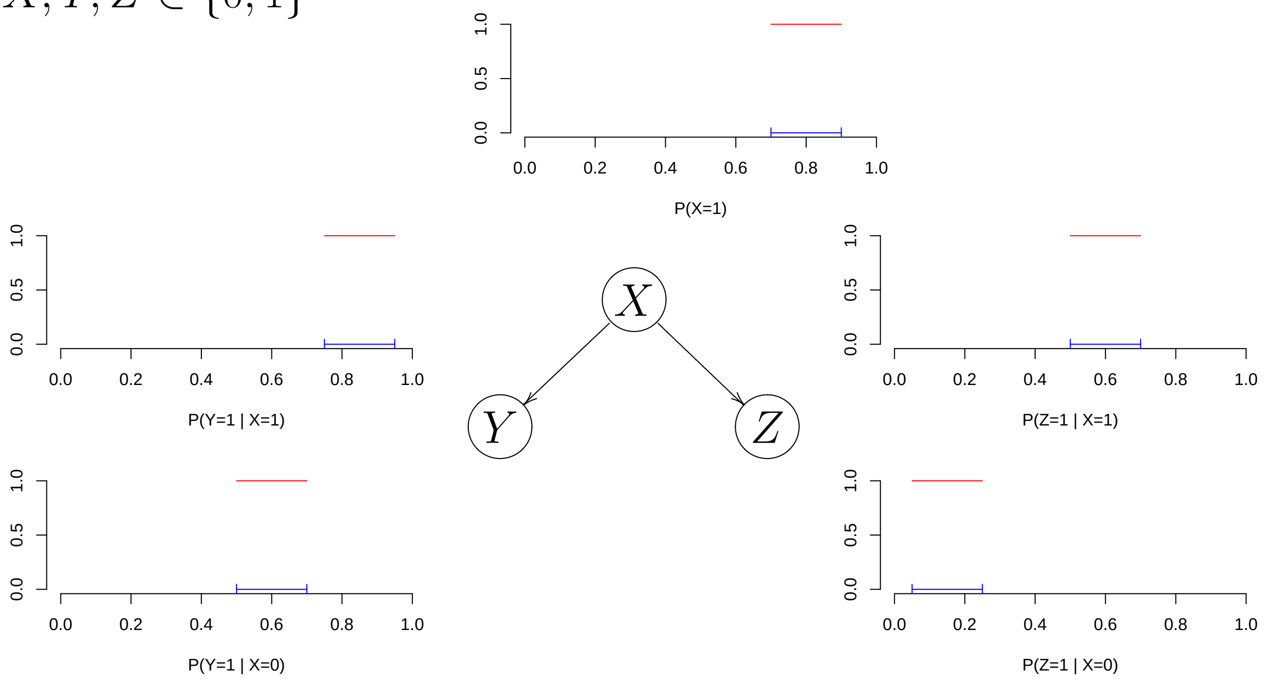
profile likelihood function $lik_g : \mathcal{G} \to (0\,,\infty)$ with

$$lik_g(\gamma) \propto \sup_{P \in \mathcal{P}\,:\,g(P)=\gamma} lik(P) \propto \sup_{\mu \in \mathcal{M}\,:\,g\left(\frac{\mu}{\mu(\Omega)}\right)=\gamma} \mu(\Omega)$$

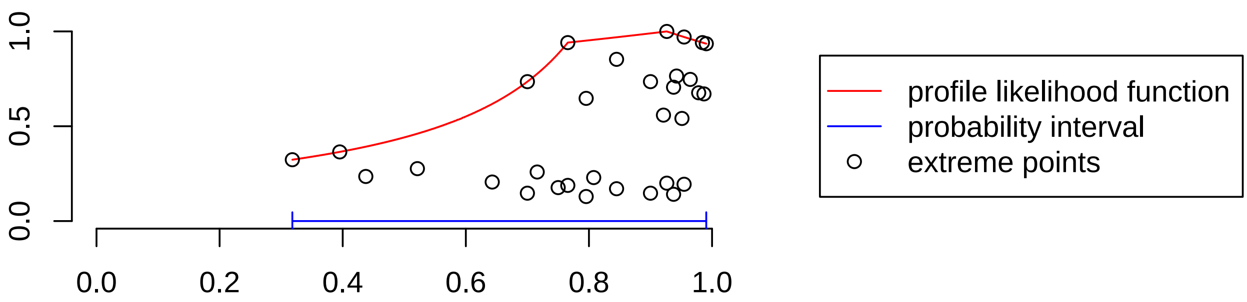
in particular (**Theorem 3**): if $\mathcal{M} = \mathrm{ch}(\{lik(P_i)\,P_i : i \in \{1,\ldots,m\}\})$, then the profile likelihood function of the expected value $E_P(X)$ of a random variable X is **piecewise hyperbolic** and determined by the pairs $(E_{P_i}(X),\, lik(P_i))$ with $i \in \{1,\ldots,m\}$

example of piecewise hyperbolic profile likelihood function:

$X, Y, Z \in \{0, 1\}$

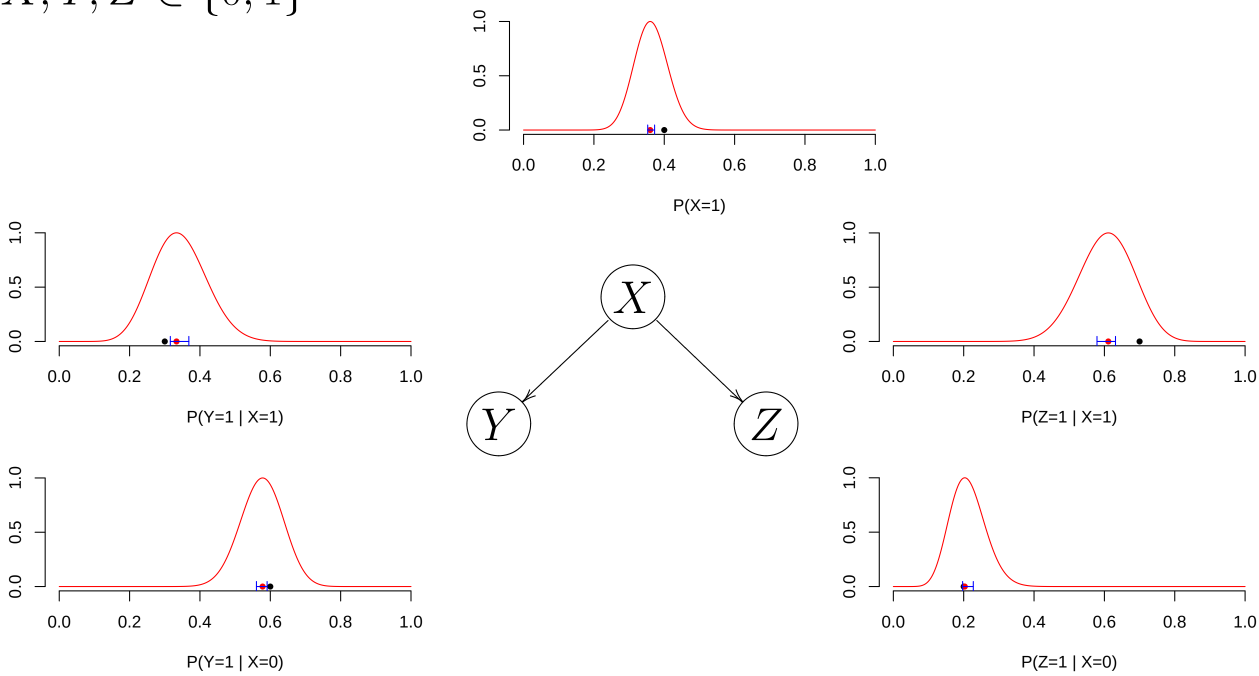


$\Rightarrow \;\; P(X = 1 \,|\, Y = 0,\; Z = 1)$:

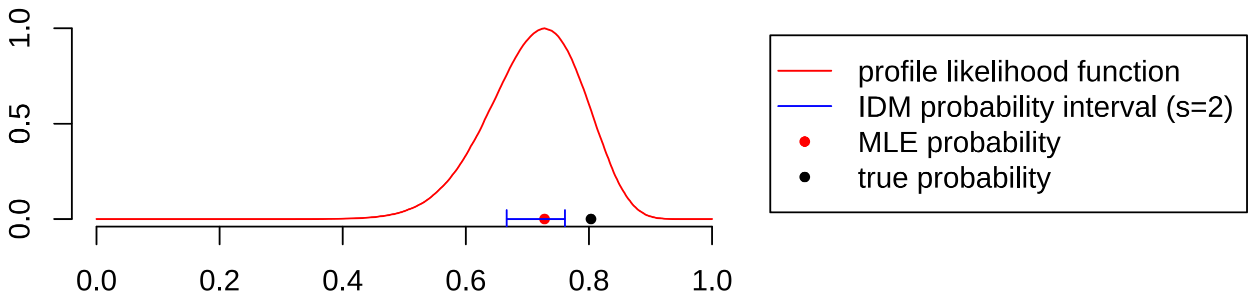


example with training data:

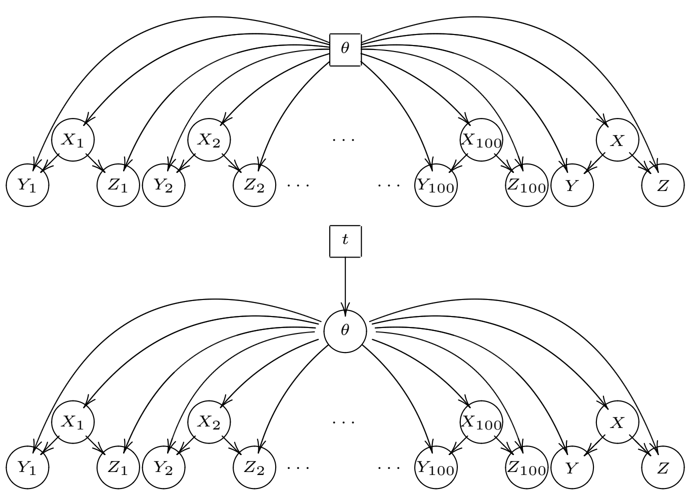
$X, Y, Z \in \{0, 1\}$



$\Rightarrow \; P(X = 1 \,|\, Y = 0,\; Z = 1) :$



global models (without and with IDMs):



$\theta, t \in [0, 1]^5$

prior ignorance about the root, and precise (conditional) probabilities about the other nodes