Combining Belief Functions Issued from Dependent Sources

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Definition. A piece of information about a finite set of propositional variables is described by a *basic belief assignment* (*bba*)

$$m : 2^\Omega \to [0,1] \quad \text{s.t.} \quad m(\emptyset) = 0 \quad \text{and} \quad \sum_A m(A) = 1,$$

where Ω is the set of valuations of the propositional language (i.e. Ω is the set of "possible worlds" $\Rightarrow$ the "open-world assumption" does not make sense).
The respective *belief* and *plausibility functions* on Ω are defined by

$$bel(A) = \sum_{B \subseteq A} m(B) \quad \text{and} \quad pl(A) = \sum_{B \cap A \neq \emptyset} m(B).$$

to pool the information issued from two sources
$\rightsquigarrow$ combine the respective bbas m_1 and m_2 in a new bba m_{12}

independence of the sources assumed
(this assumption can be justified only by analogies with other situations in which it proved to be sensible)
$\rightsquigarrow$ use Dempster's rule of combination:

$$m_{12}(A) \propto \sum_{B \cap C = A} m_1(B)\, m_2(C) \quad \text{if } A \neq \emptyset$$

Generalized Combination Rule

to allow the dependence of the sources
$\rightsquigarrow$ generalize Dempster's rule

Definition. A *joint belief assignment* (*jba*) with marginal bbas m_1 and m_2 is a function

$$\underline{m} : 2^\Omega \times 2^\Omega \to [0,1] \quad \text{s.t.} \quad \sum_B \underline{m}(A,B) = m_1(A)$$
$$\text{and} \quad \sum_A \underline{m}(A,B) = m_2(B).$$

combination with respect to a jba $\underline{m}$:

$$m_{12}(A) \propto \sum_{B \cap C = A} \underline{m}(B,C) \quad \text{if } A \neq \emptyset$$

($\Rightarrow$ the independence assumption corresponds to the choice of the jba $\underline{m}(A,B) = m_1(A)\, m_2(B)$)

nothing assumed about the sources
$\rightsquigarrow$ play safe and choose the "most conservative" combination

Minimal Conflict

Definition. A combination bel_{12} of two belief functions bel_1 and bel_2 is *monotonic* if

$$bel_{12} \geq bel_1 \quad \text{and} \quad bel_{12} \geq bel_2.$$

Definition. The *conflict* of the combination with respect to a jba $\underline{m}$ is

$$\sum_{A \cap B = \emptyset} \underline{m}(A, B).$$

(no conflict $\Rightarrow$ the combination is monotonic)

the conflict is a good index for the nonmonotonicity of a combination
$\leadsto$ the "most conservative" combination has minimal conflict

Theorem. *The* ***minimal conflict*** *of the combinations of* bel_1 *and* bel_2 *is*

$$\max_A \big(bel_1(A) - pl_2(A) \big). \tag{1}$$

Corollary. *The monotonicity of the combination of* bel_1 *and* bel_2 *is admissible (i.e.* $\exists$ *bel s.t. bel* $\geq bel_1$ *and bel* $\geq bel_2$*) if and only if they are compatible (i.e.* $bel_1 \leq pl_2$*).*
In this case, ***the combinations with minimal conflict are monotonic****.*

Generalized Bayes' Theorem

In the generalized Bayes' theorem, combinations with minimal conflict lead to better results than combinations obtained from Dempster's rule.

Consider n hypotheses $h_1, \ldots, h_n$ implying the belief functions $bel_1, \ldots, bel_n$ on Ω, respectively.
Let the belief function bel_o on Ω represent an observation and let $c_1, \ldots, c_n$ be the conflicts of its combination with $bel_1, \ldots, bel_n$, respectively.
In the simplest case, the prior belief function on $\{h_1, \ldots, h_n\}$ is an epistemic probability $p_1, \ldots, p_n$. In this case, the posterior belief function is the epistemic probability $p'_1, \ldots, p'_n$, with

$$p'_i \propto (1 - c_i)\, p_i.$$

Thus the conflicts come out as the measure of the disagreement between the respective hypotheses and the observation.

If the c_i are the minimal conflicts, then from $bel_o \leq pl_i$ (i.e. h_i is compatible with the observation) follows $p'_i \geq p_i$.
This is not assured if we use Dempster's rule: $p'_i < p_i$ is possible even if $bel_o = bel_i$ (i.e. h_i is "perfect").

As a measure of the disagreement between two belief functions, the minimal conflict (1) is much better than the conflict of Dempster's rule.

Example. $\Omega = \{a, b\}$, $n = 4$, $bel_o = bel_1$, bel_2 is vacuous.

				Dempster's rule		minimal conflict	
i	$m_i(\{a\})$	$m_i(\{b\})$	$m_i(\Omega)$	c_i	p'_i/p_i	c_i	p'_i/p_i
1	0.5	0.4	0.1	0.4	0.84	0	1.18
2	0	0	1	0	1.40	0	1.18
3	0	0.7	0.3	0.35	0.91	0.2	0.94
4	1	0	0	0.4	0.84	0.4	0.71

Minimal Specificity

Definition. The *measure of nonspecificity* of a belief function with bba m is

$$\sum_{A \neq \emptyset} m(A) \log_2 |A|.$$

if the combination with minimal conflict is not unique
$\rightsquigarrow$ the "most conservative" combination has minimal specificity (i.e. it maximizes the measure of nonspecificity) among the ones with minimal conflict

Definition. bel_2 is a *specialization* of bel_1 if m_2 can be obtained through redistribution of $m_1(A)$ to the non-empty sets $B \subseteq A$, for all $A \subseteq \Omega$.

Theorem. *bel_1 and bel_2 have a common specialization if and only if they are compatible (i.e. $bel_1 \leq pl_2$).*
In this case, ***the combinations with minimal specificity among the ones with minimal conflict are the least specific common specializations*** *of bel_1 and bel_2.*

to obtain a combination with minimal specificity among the ones with minimal conflict
$\rightsquigarrow$ maximize a linear functional on the convex polytope (in $\mathbb{R}^{2^{2|\Omega|}}$) of the jbas

the solutions build a convex polytope
$\rightsquigarrow$ choose a point of the polytope in such a way that the obtained rule $(bel_1, bel_2) \mapsto bel_1 \odot bel_2$ satisfies some requirements of invariance (choose for instance the centre of the polytope)

Conservative Combination Rule

The obtained "most conservative" combination rule $\odot$ has the following properties.

- **commutativity**:
$$bel_1 \odot bel_2 = bel_2 \odot bel_1$$

- **monotonicity** (if admissible, i.e. if $\exists$ bel s.t. $bel \geq bel_1$ and $bel \geq bel_2$):
$$bel_1 \odot bel_2 \geq bel_1 \quad \text{and} \quad bel_1 \odot bel_2 \geq bel_2$$

- $bel_1 \odot bel_2$ is a **least specific common specialization** of bel_1 and bel_2 (if a common specialization exists)

  $\Rightarrow$ **absorption**:
$$bel_s \text{ is a specialization of } bel \quad \Rightarrow \quad bel_s \odot bel = bel_s$$

  $\Rightarrow$ **idempotency**:
$$bel \odot bel = bel$$

But minimization of conflict and idempotency are both incompatible with associativity.

Thus the binary rule $\odot$ is not associative, but it can be easily extended to an n-ary rule for the simultaneous combination of any number of belief functions: simply consider the n-dimensional jbas instead of the 2-dimensional ones.