Maxitive Integral of Real-Valued Functions

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Abstract. The paper pursues the definition of a maxitive integral on all real-valued functions (i.e., the integral of the pointwise maximum of two functions must be the maximum of their integrals). This definition is not determined by maxitivity alone: additional requirements on the integral are necessary. The paper studies the consequences of additional requirements of invariance with respect to affine transformations of the real line.

Keywords: maxitive measures, nonadditive integrals, location and scale invariance, Shilkret integral, convexity, subadditivity.

1 Introduction

In theories of reasoning and decision making under uncertainty, measures (or capacities) are used to describe uncertain belief or information, and can be extended to integrals in order to evaluate and compare uncertain (real-valued) payoffs. In particular, the additive capacities used in the Bayesian theory can be extended (almost) uniquely to an additive integral. By contrast, the extension to a maxitive integral of the maxitive capacities used in alternative theories is not unique, and additional requirements are needed in order to determine it. The present paper focuses on additional requirements of invariance with respect to the choice of the measurement scale of the payoffs.

The invariance with respect to the choice of the scale unit determines the Shilkret integral on nonnegative functions. This integral satisfies some properties that are important for the evaluation of uncertain payoffs, such as subadditivity or the law of iterated expectations, but it cannot be extended in a reasonable way to a maxitive integral on all real-valued functions. By contrast, the invariance with respect to the choice of the zero point of the measurement scale (almost) determines a maxitive integral on all real-valued functions, called convex integral. The name comes from the property of convexity, which is satisfied besides other important ones for the evaluation of uncertain payoffs, such as the law of iterated expectations.

The paper is organized as follows. The next section introduces the concepts of capacities and of integrals as their extensions. The Shilkret and convex integrals are then studied in Sects. 3 and 4, respectively. The final section gives directions for further research.

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2 Integrals as Extensions of Capacities

Let Ω be a set and let μ be a capacity on Ω. That is, $\mu : \mathcal{P}(\Omega) \to [0,1]$ is a monotonic set function such that $\mu(\varnothing) = 0$ and $\mu(\Omega) = 1$, where monotonic means that $\mu(A) \leq \mu(B)$ for all $A \subseteq B \subseteq \Omega$.

The capacity μ can be interpreted as a quantitative description of uncertain belief or information about $\omega \in \Omega$. The larger the value $\mu(A)$, the larger the plausibility of $\omega \in A$, or the larger the implausibility of $\omega \notin A$. This is in agreement with the monotonicity of μ, while the requirements $\mu(\varnothing) = 0$ and $\mu(\Omega) = 1$ can be interpreted as meaning that $\omega \in \varnothing$ is impossible and that nothing speaks against $\omega \in \Omega$, respectively.

More precise interpretations of the values of μ can lead to additional requirements on the set function μ. The best known additional requirement is (finite) additivity: $\mu(A \cup B) = \mu(A) + \mu(B)$ for all disjoint $A, B \subseteq \Omega$. Additive capacities (also called probability charges, or finitely additive probability measures) are the quantitative descriptions of uncertain belief about $\omega \in \Omega$ used in the Bayesian theory [1,2].

The continuity (from below and from above) of the set function μ is often required together with the additivity for technical reasons. The resulting requirement is countable additivity: $\mu(\bigcup_{n \in \mathbb{N}} A_n) = \sum_{n \in \mathbb{N}} \mu(A_n)$ for all sequences $(A_n)_{n \in \mathbb{N}}$ of (pairwise) disjoint $A_n \subseteq \Omega$. Countably additive capacities (also called probability measures) are the quantitative descriptions of uncertain information about $\omega \in \Omega$ used in probability theory [3]. However, countable additivity is too strong a requirement when Ω is uncountable, and therefore μ cannot usually be defined on the whole power set $\mathcal{P}(\Omega)$, at least under the axiom of choice [4].

A common requirement on the set function μ alternative to additivity is (finite) maxitivity: $\mu(A \cup B) = \mu(A) \vee \mu(B)$ for all (disjoint) $A, B \subseteq \Omega$. Maxitive capacities (also called possibility measures, consonant plausibility functions, or idempotent measures) have been studied in various contexts [5,6,7]. As quantitative descriptions of uncertain belief or information about $\omega \in \Omega$, maxitive capacities play a central role in possibility theory [8], but they also appear for instance as consonant plausibility functions in the theory of belief functions [9], or as supremum preserving upper probabilities in the theory of imprecise probabilities [10]. Moreover, the description of uncertain belief by means of maxitive capacities also corresponds for example to the descriptions by means of degrees of potential surprise [11], or of degrees of support by eliminative induction [12]. Of particular importance in statistical applications is the fact that the likelihood (ratio) of composite hypotheses is a maxitive capacity [13,14].

The requirement of maxitivity of the set function μ can be generalized to the requirement of κ-maxitivity (where κ is a cardinal): $\mu(\bigcup_{A \in \mathcal{A}} A) = \bigvee_{A \in \mathcal{A}} \mu(A)$ for all nonempty $\mathcal{A} \subseteq \mathcal{P}(\Omega)$ of cardinality at most κ. Maxitivity corresponds to κ-maxitivity when κ is finite and at least 2. Contrary to the case of additivity, the requirement of κ-maxitivity for infinite cardinals κ does not pose any problem: a κ-maxitive set function can always be extended from a ring of subsets of Ω to the whole power set $\mathcal{P}(\Omega)$ [15, Theorem 1].

The capacity μ on Ω has a particularly simple description when it is κ-maxitive with κ the cardinality of Ω. In fact, μ is then completely described by its values on the singletons: $\mu(A) = \bigvee_{\omega \in A} \mu\{\omega\}$ for all nonempty $A \subseteq \Omega$. This implies in particular the κ-maxitivity of μ for all cardinals κ, also called complete maxitivity. For example, in statistics, the likelihood of composite hypotheses is a completely maxitive capacity: $\lambda(\mathcal{H}) = \bigvee_{\theta \in \mathcal{H}} \lambda\{\theta\}$ for all composite hypotheses $\mathcal{H} \subseteq \Theta$, where Θ is a set of simple hypotheses, and $\theta \mapsto \lambda\{\theta\}$ is the (relative) likelihood function on Θ [13,14].

Since each $A \subseteq \Omega$ can be identified with its indicator function I_A on Ω, the capacity μ can be identified with a functional on the set of all indicator functions I_A with $A \subseteq \Omega$. The remaining of this paper studies extensions of this functional to larger classes of functions on Ω, and in order to avoid trivial results, it is assumed that there is a $C \subseteq \Omega$ such that $0 < \mu(C) < 1$.

Let $\mathcal{F}$ be a set of extended real-valued functions $f : \Omega \to \overline{\mathbb{R}}$. A functional $F : \mathcal{F} \to \overline{\mathbb{R}}$ is said to extend the capacity μ to $\mathcal{F}$ if $F(I_A) = \mu(A)$ for all $A \subseteq \Omega$. In this definition, as in the rest of the paper, it is assumed as part of the condition that the expressions are well-defined. That is, F can extend μ to $\mathcal{F}$ only if $I_A \in \mathcal{F}$ for all $A \subseteq \Omega$, because otherwise the expression $F(I_A) = \mu(A)$ would not be well-defined.

2.1 Extension of Additive Capacities

In the Bayesian theory, the uncertain belief about $\omega \in \Omega$ is described by an additive capacity μ on Ω, while the evaluation of a (bounded) uncertain payoff $f(\omega)$ on the basis of this belief is given by its expectation $\int f \,\mathrm{d}\mu$, which is defined as follows.

Let $\mathcal{B}$ be the set of all bounded functions $f : \Omega \to \mathbb{R}$, and let $\mathcal{S} \subseteq \mathcal{B}$ be the subset of all simple functions (i.e., all functions $s : \Omega \to \mathbb{R}$ such that their images $s(\Omega)$ are finite). The (standard) integral of $f \in \mathcal{B}$ with respect to an additive capacity μ on Ω is denoted by $\int f \,\mathrm{d}\mu$ and is defined as

$$\int f \,\mathrm{d}\mu = \bigvee_{s \in \mathcal{S} \,:\, s \leq f} \; \sum_{x \in s(\Omega)} x \, \mu\{s = x\} = \bigwedge_{s \in \mathcal{S} \,:\, s \geq f} \; \sum_{x \in s(\Omega)} x \, \mu\{s = x\},$$

where $\{s = x\}$ is the usual short form for the set $\{\omega \in \Omega : s(\omega) = x\}$. The integral is well-defined when μ is additive [16], and corresponds to the Lebesgue integral when μ is countably additive [17].

The next theorem shows that the integral with respect to an additive capacity μ on Ω is the unique monotonic, additive extension of μ to $\mathcal{B}$. A functional $F : \mathcal{F} \to \overline{\mathbb{R}}$ is said to be monotonic if $F(f) \leq F(g)$ for all $f, g \in \mathcal{F}$ such that $f \leq g$, while F is said to be (finitely) additive if $F(f + g) = F(f) + F(g)$ for all $f, g \in \mathcal{F}$. Note that the additivity of a functional F on $\mathcal{B}$ implies its monotonicity when some weak additional requirement is satisfied: for example when $F(f) \geq 0$ for all $f \in \mathcal{B}$ such that $f \geq 0$.

Theorem 1. *When μ is additive, its additive extension to $\mathcal{B}$ is not unique, but the functional $f \mapsto \int f \, \mathrm{d}\mu$ on $\mathcal{B}$ is the unique monotonic, additive extension of μ to $\mathcal{B}$.*

Proof. When μ is additive, the functional $F : f \mapsto \int f \, \mathrm{d}\mu$ extends μ and is monotonic and additive [16, Chap. 4]. If F' is an additive extension of μ to $\mathcal{B}$, then its additivity implies $F'(\alpha f) = \alpha F'(f)$ for all $\alpha \in \mathbb{Q}$ and all $f \in \mathcal{B}$, and therefore also

$$F'(s) = \sum_{x \in s(\Omega)} x \, \mu\{s = x\} = F(s)$$

for all simple functions $s \in \mathcal{S}$ such that $s(\Omega) \subseteq \mathbb{Q}$. If F' is also monotonic, then its monotonicity implies $F'(f) = F(f)$ for all $f \in \mathcal{B}$.

However, additive extensions of μ to $\mathcal{B}$ that are not monotonic also exist, at least under the axiom of choice. Let $\tau : \mathbb{R} \to \mathbb{R}$ be a discontinuous additive function such that $\tau(0) = 0$ and $\tau(1) = 1$ [18, Corollary 5.2.1]. Then the functional $F' : f \mapsto \int \tau \circ f \, \mathrm{d}\mu$ on $\mathcal{B}$ is an additive extension of μ, but $F' \neq F$. □

2.2 Extension of Maxitive Capacities

In the Bayesian theory, the uncertain belief about $\omega \in \Omega$ is described by an additive capacity μ on Ω, and the evaluations of uncertain payoffs $f \in \mathcal{B}$ are described by the unique monotonic, additive extension of μ to $\mathcal{B}$. Analogously, when the uncertain belief or information about $\omega \in \Omega$ is described by a maxitive capacity μ on Ω, the evaluations of uncertain payoffs $f \in \mathcal{B}$ can be described by a maxitive extension of μ to $\mathcal{B}$. However, the next theorem shows that the maxitive extension to $\mathcal{B}$ of a maxitive capacity μ on Ω is not unique. A functional $F : \mathcal{F} \to \overline{\mathbb{R}}$ is said to be maxitive if $F(f \vee g) = F(f) \vee F(g)$ for all $f, g \in \mathcal{F}$. Note that the maxitivity of a functional implies its monotonicity.

Theorem 2. *When μ is maxitive, its maxitive extension to $\mathcal{B}$ is not unique.*

Proof. When μ is maxitive, both functionals

$$F : f \mapsto \bigvee_{x \in \mathbb{R}_{>0}} x \, \mu\{f > x\} \quad \text{and} \quad F' : f \mapsto \bigvee_{x \in \mathbb{R}_{>0}} (x \wedge \mu\{f > x\})$$

on $\mathcal{B}$ are maxitive extensions of μ, because $\mu\{f \vee g > x\} = \mu\{f > x\} \vee \mu\{g > x\}$ for all $f, g \in \mathcal{B}$ and all $x \in \mathbb{R}$. However, $F \neq F'$, since for instance $F(2) = 2$, while $F'(2) = 1$. When $f \geq 0$, the values $F(f)$ and $F'(f)$ are known as Shilkret and Sugeno integrals of f with respect to μ, respectively [5,19]. □

In order to obtain a unique extension to $\mathcal{B}$ of a maxitive capacity μ on Ω, additional requirements are necessary, besides maxitivity (and monotonicity). A particularly important requirement for evaluations of uncertain payoffs is their invariance with respect to changes in the measurement scale of the payoffs, such as changes in the location of the zero point or changes in the scale unit. A functional $F : \mathcal{F} \to \overline{\mathbb{R}}$ is said to be location invariant if $F(f + \alpha) = F(f) + \alpha$ for all $f \in \mathcal{F}$ and all $\alpha \in \mathbb{R}$, while F is said to be scale invariant if $F(\alpha\, f) = \alpha\, F(f)$ for all $f \in \mathcal{F}$ and all $\alpha \in \mathbb{R}_{>0}$.

The (standard) integral with respect to additive capacities is location and scale invariant [16]. The best known location and scale invariant integral with respect to nonadditive capacities is the one of Choquet [20,21]. The Choquet integral of $f \in \mathcal{B}$ with respect to a capacity μ on Ω is denoted by $\int^{\mathrm{C}} f \,\mathrm{d}\mu$ and is defined as

$$\int^{\mathrm{C}} f \,\mathrm{d}\mu = \int_{-\infty}^{0} \left(\mu\{f > x\} - 1\right) \mathrm{d}x + \int_{0}^{+\infty} \mu\{f > x\} \,\mathrm{d}x,$$

where the right-hand side is the well-defined sum of two improper Riemann integrals. The Choquet integral with respect to a capacity μ on Ω is a monotonic extension of μ to $\mathcal{B}$, which is additive when μ is additive [22]. Therefore, $\int^{\mathrm{C}} f \,\mathrm{d}\mu = \int f \,\mathrm{d}\mu$ for all $f \in \mathcal{B}$ when μ is additive.

The next theorem shows that no maxitive extension to $\mathcal{B}$ of a maxitive capacity μ on Ω is location and scale invariant. Maxitive extensions satisfying one of these two additional requirements are studied in the next two sections.

Theorem 3. *When μ is maxitive, there is no location and scale invariant, maxitive extension of μ to $\mathcal{B}$.*

Proof. Let F be a scale invariant, maxitive extension to $\mathcal{B}$ of a maxitive capacity μ on Ω. As assumed above, there is a $C \subseteq \Omega$ such that $0 < \mu(C) < 1$. Hence, $\mu(\Omega \setminus C) = 1$ and

$$F(I_C + 1) = F\left((2\, I_C) \vee I_{\Omega \setminus C}\right) = (2\, \mu(C)) \vee 1 < \mu(C) + 1 = F(I_C) + 1,$$

and therefore F is not location invariant. $\square$

3 Shilkret Integral

Let $\mathcal{E}$ be the set of all extended real-valued functions $f : \Omega \to \overline{\mathbb{R}}$, let $\mathcal{E}^+ \subseteq \mathcal{E}$ be the subset of all nonnegative functions, and let $\mathcal{B}^+ = \mathcal{B} \cap \mathcal{E}^+$ the subset of all bounded, nonnegative functions. The Shilkret integral of $f \in \mathcal{E}^+$ with respect to a capacity μ on Ω is denoted by $\int^{\mathrm{S}} f \,\mathrm{d}\mu$ and is defined as

$$\int^{\mathrm{S}} f \,\mathrm{d}\mu = \bigvee_{x \in \mathbb{R}_{>0}} x\, \mu\{f > x\}.$$

The Shilkret integral has a particularly simple expression when μ is completely maxitive: $\int^{\mathrm{S}} f \,\mathrm{d}\mu = \bigvee_{\omega \in \Omega} f(\omega)\, \mu\{\omega\}$ for all $f \in \mathcal{E}^+$ [5].

The next theorem shows that the Shilkret integral with respect to a maxitive capacity μ on Ω is the unique scale invariant, maxitive extension of μ to $\mathcal{B}^+$. The Shilkret integral maintains κ-maxitivity also for infinite cardinals κ. A functional $F : \mathcal{F} \to \overline{\mathbb{R}}$ is said to be κ-maxitive if $F(\bigvee_{f \in \mathcal{G}} f) = \bigvee_{f \in \mathcal{G}} F(f)$ for all nonempty $\mathcal{G} \subseteq \mathcal{F}$ of cardinality at most κ.

Theorem 4. *When μ is maxitive, the functional $f \mapsto \int^{\mathrm{S}} f \,\mathrm{d}\mu$ on $\mathcal{B}^+$ is the unique scale invariant, maxitive extension of μ to $\mathcal{B}^+$. Moreover, when κ is an infinite cardinal and μ is κ-maxitive, the functional $f \mapsto \int^{\mathrm{S}} f \,\mathrm{d}\mu$ on $\mathcal{E}^+$ is the unique scale invariant, κ-maxitive extension of μ to $\mathcal{E}^+$.*

Proof. When κ is a cardinal and μ is κ-maxitive, the functional $f \mapsto \int^{\mathrm{S}} f \,\mathrm{d}\mu$ on $\mathcal{E}^+$ is a scale invariant, κ-maxitive extension of μ to $\mathcal{E}^+$ [15, Lemma 1]. Such an extension is unique on $\mathcal{B}^+$ when $\kappa \geq 2$ [15, Theorem 2 (iii)], and it is unique also on $\mathcal{E}^+$ when κ is infinite [15, Theorem 3 (iii)]. □

An important property for evaluations of uncertain payoffs is convexity, meaning that diversification does not increase the risk [23,24]. A functional $F : \mathcal{F} \to \overline{\mathbb{R}}$ is said to be convex if $F(\lambda f + (1-\lambda)\, g) \leq \lambda F(f) + (1-\lambda)\, F(g)$ for all $\lambda \in (0,1)$ and all $f, g \in \mathcal{F}$, whereas F is said to be subadditive if $F(f+g) \leq F(f) + F(g)$ for all $f, g \in \mathcal{F}$. Note that convexity and subadditivity are equivalent for a scale invariant functional.

The characterization of the capacities with respect to which the Choquet integral is convex (i.e., subadditive) is a well-known result [20,22,21]. The next theorem gives also a characterization of the capacities with respect to which the Shilkret integral is convex (i.e., subadditive). The capacity μ is said to be submodular if $\mu(A \cup B) + \mu(A \cap B) \leq \mu(A) + \mu(B)$ for all $A, B \subseteq \Omega$. Note that additive or maxitive capacities are submodular.

Theorem 5. *The functional $f \mapsto \int^{\mathrm{C}} f \,\mathrm{d}\mu$ on $\mathcal{B}$ is convex if and only if μ is submodular, while the functional $f \mapsto \int^{\mathrm{S}} f \,\mathrm{d}\mu$ on $\mathcal{E}^+$ is convex if and only if μ is maxitive.*

Proof. Both functionals $f \mapsto \int^{\mathrm{C}} f \,\mathrm{d}\mu$ on $\mathcal{B}$ and $f \mapsto \int^{\mathrm{S}} f \,\mathrm{d}\mu$ on $\mathcal{E}^+$ are scale invariant. The first one is subadditive if and only if μ is submodular [21, Chap. 6], while the second one is subadditive if and only if μ is maxitive [15, Theorem 4 (iii)]. □

The Shilkret integral with respect to maxitive capacities satisfies also other important properties for evaluations of uncertain payoffs, such as the law of iterated expectations (or evaluations). By contrast, the Choquet integral satisfies this law only with respect to additive capacities (i.e., only when it corresponds to the standard integral) [15,25]. However, the Shilkret integral is defined only for nonnegative functions, and its extension to functions taking negative values is problematic.

When μ is maxitive, the Shilkret integral is the unique scale invariant, maxitive extension of μ to $\mathcal{B}^+$, but the next theorem shows that its further (scale invariant, maxitive) extension to $\mathcal{B}$ is not unique. In fact, the values assigned to negative functions by such extensions are independent from μ. To impose a dependence from μ, some kind of symmetry of the extension could be required. For example, since μ determines the values of its extensions on all indicator functions I_A, the determination of the values on all negative indicator functions $-I_A$ could be required. An extension $F : \mathcal{F} \to \overline{\mathbb{R}}$ of a capacity μ on Ω is said to be symmetric if $F(-I_A) = -\mu(A)$ for all $A \subseteq \Omega$, while F is said to be dual symmetric if $F(-I_A) = -\overline{\mu}(A)$ for all $A \subseteq \Omega$, where the dual capacity $\overline{\mu}$ on Ω is defined by $\overline{\mu}(A) = 1 - \mu(\Omega \setminus A)$. Note that all location invariant extensions of a capacity are dual symmetric, and the (standard) integral with respect to an additive capacity is also symmetric.

However, the next theorem also shows that no scale invariant, maxitive extension to $\mathcal{B}$ of a maxitive capacity μ on Ω is symmetric or dual symmetric, and neither is it convex and calibrated. A functional $F : \mathcal{F} \to \overline{\mathbb{R}}$ is said to be calibrated if $F(\alpha) = \alpha$ for all constant functions $\alpha \in \mathcal{F}$. Note that all scale invariant extensions of μ to $\mathcal{B}^+$, all scale invariant, (dual) symmetric extensions of μ to $\mathcal{B}$, and all location invariant extensions of μ to $\mathcal{B}$ are calibrated.

Theorem 6. *When μ is maxitive and γ is a real-valued function on Ω such that $\bigwedge_{\omega \in \Omega} \gamma(\omega) = 1$, the functional*

$$f \mapsto \begin{cases} \bigvee_{\omega \in \Omega} f(\omega)\, \gamma(\omega) & \text{if } f < 0, \\ \int^{\mathrm{S}} (f \vee 0)\, \mathrm{d}\mu & \text{otherwise} \end{cases}$$

on $\mathcal{E}$ is a scale invariant, calibrated, maxitive extension of μ to $\mathcal{E}$, but no scale invariant, calibrated, maxitive extension of μ to $\mathcal{B}$ is symmetric, dual symmetric, or convex.

Proof. Since the functional $f \mapsto \int^{\mathrm{S}} f\, \mathrm{d}\mu$ on $\mathcal{E}^+$ is a scale invariant, calibrated, maxitive extension of μ to $\mathcal{E}^+$ when μ is maxitive, its further extension to $\mathcal{E}$ defined in the theorem is also scale invariant, calibrated, and maxitive.

Let F be a scale invariant, calibrated, maxitive extension to $\mathcal{B}$ of a maxitive capacity μ on Ω. As assumed above, there is a $C \subseteq \Omega$ such that $0 < \mu(C) < 1$. Hence, $\mu(\Omega \setminus C) = 1$ and

$$F(-I_C) \vee F(-I_{\Omega \setminus C}) = 0 > -\mu(C) = (-\mu(C)) \vee (-\mu(\Omega \setminus C)),$$

and therefore F is not symmetric. Neither can F be dual symmetric, because $0 < \overline{\mu}(\Omega \setminus C) < 1$, while

$$F(-I_{\Omega \setminus C}) = F\big((-2\, I_{\Omega \setminus C}) \vee (-1)\big) = \big(2\, F(-I_{\Omega \setminus C})\big) \vee (-1)$$

implies $F(-I_{\Omega \setminus C}) \in \{-1, 0\}$. Finally, if $f = I_C \vee \mu(C)$, then $F(f) = \mu(C)$ and since

$$F\left(f + (-\mu(C))\right) = F\left((1 - \mu(C))\, I_C\right) > 0 = F(f) + F\left(-\mu(C)\right),$$

F is not subadditive (i.e., convex). □

4 Convex Integral

The convex integral of $f \in \mathcal{E}$ with respect to a capacity μ on Ω is denoted by $\int^{\mathrm{X}} f\, \mathrm{d}\mu$ and is defined as

$$\int^{\mathrm{X}} f\, \mathrm{d}\mu = \bigvee_{x \in \mathbb{R}} \left(x + \tau \circ \mu\{f > x\}\right),$$

where τ is the function on $[0,1]$ defined by $\tau(0) = -\infty$ and $\tau(x) = x - 1$ otherwise. The convex integral has a particularly simple expression when μ is completely maxitive: $\int^{\mathrm{X}} f \,\mathrm{d}\mu = \bigvee_{\omega \in \Omega \,:\, \mu\{\omega\} > 0} (f(\omega) + \mu\{\omega\} - 1)$ for all $f \in \mathcal{E}$.

The next theorem shows that the convex integral with respect to a maxitive capacity μ on Ω is the unique location invariant, maxitive extension of μ to $\mathcal{B}$, when $\varnothing$ is the only null set (i.e., $\mu(A) > 0$ for all nonempty $A \subseteq \Omega$). When there are nonempty null sets, the location invariant, maxitive extension to $\mathcal{B}$ of a maxitive capacity μ on Ω is not unique, but the convex integral is the only null preserving one. An extension $F : \mathcal{F} \to \overline{\mathbb{R}}$ of a capacity μ on Ω is said to be null preserving if $F(f) = 0$ for all $f \in F$ such that $\mu\{f \neq 0\} = 0$. Note that all extensions of a capacity are null preserving when $\varnothing$ is the only null set.

Theorem 7. *When μ is maxitive, the functional $f \mapsto \int^{\mathrm{X}} f \,\mathrm{d}\mu$ on $\mathcal{B}$ is the unique location invariant, maxitive extension of μ to $\mathcal{B}$ if and only if $\varnothing$ is the only null set, and in general it is the unique location invariant, null preserving, maxitive extension of μ to $\mathcal{B}$. Moreover, when κ is an infinite cardinal and μ is κ-maxitive, the functional $f \mapsto \int^{\mathrm{X}} f \,\mathrm{d}\mu$ on $\mathcal{E}$ is the unique location invariant, null preserving, κ-maxitive extension of μ to $\mathcal{E}$.*

Proof. When $\kappa \geq 2$ is a cardinal and μ is κ-maxitive, the functional $f \mapsto \int^{\mathrm{X}} f \,\mathrm{d}\mu$ on $\mathcal{E}$ is a location invariant, null preserving, κ-maxitive extension of μ to $\mathcal{E}$ [15, Corollary 5]. Such an extension is unique on $\mathcal{B}$ [15, Corollary 6], and it is unique also on $\mathcal{E}$ when κ is infinite [15, Corollary 7].

Let ν be the set function on $\mathcal{P}(\Omega)$ defined by $\nu(\varnothing) = -\infty$ and $\nu(A) = \mu(A) - 1$ otherwise. When μ is maxitive, the functional $f \mapsto \bigvee_{x \in \mathbb{R}} (x + \nu\{f > x\})$ on $\mathcal{B}$ is a location invariant, maxitive extension of μ to $\mathcal{B}$ [15, Corollary 6], and it differs from the functional $f \mapsto \int^{\mathrm{X}} f \,\mathrm{d}\mu$ on $\mathcal{B}$ when there are nonempty null sets. □

Convexity and subadditivity are not equivalent for functionals that are not scale invariant. The next theorem shows that the convex integral with respect to maxitive capacities is not subadditive. However, it is convex, and this is the reason for its name.

Theorem 8. *The functional $f \mapsto \int^{\mathrm{X}} f \,\mathrm{d}\mu$ on $\mathcal{B}$ is convex if and only if μ is maxitive. But when μ is maxitive, no location invariant, maxitive extension of μ to $\mathcal{B}$ is subadditive.*

Proof. When μ is maxitive, the functional $f \mapsto \int^{\mathrm{X}} f \,\mathrm{d}\mu$ on $\mathcal{B}$ is convex [15, Theorem 7]. But when μ is not maxitive, there are $A, B \subseteq \Omega$ and $\alpha \in \mathbb{R}_{>0}$ such that $\mu(A \cup B) - \alpha > \mu(A) \vee \mu(B)$. Hence, if $g = I_{A \cup B} + \alpha\, I_B$ and $h = I_{A \cup B} - \alpha\, I_B$, then $\int^{\mathrm{X}} g \,\mathrm{d}\mu = \mu(A \cup B)$ and $\int^{\mathrm{X}} h \,\mathrm{d}\mu = \mu(A \cup B) - \alpha$, and since

$$\int^{\mathrm{X}} \left(\tfrac{1}{2}\, g + \tfrac{1}{2}\, h\right) \mathrm{d}\mu = \mu(A \cup B) > \tfrac{1}{2} \int^{\mathrm{X}} g \,\mathrm{d}\mu + \tfrac{1}{2} \int^{\mathrm{X}} h \,\mathrm{d}\mu,$$

the functional $f \mapsto \int^{\mathrm{X}} f \,\mathrm{d}\mu$ on $\mathcal{B}$ is not convex.

Let F be a location invariant, maxitive extension to $\mathcal{B}$ of a maxitive capacity μ on Ω. As assumed above, there is a $C \subseteq \Omega$ such that $0 < \mu(C) < 1$, and thus there is an $n \in \mathbb{N}$ such that $n\ (1 - \mu(C)) \geq 1$. Hence, if $f = (1 - \mu(C))\ I_C$, then

$$F(n\, f) \geq \mu(C) > 0 = n\ (F\,(I_C \vee \mu(C)) - \mu(C)) = n\, F(f),$$

and therefore F is not subadditive. □

Besides convexity and location invariance, the convex integral with respect to maxitive capacities satisfies also other important properties for evaluations of uncertain payoffs, such as the law of iterated expectations (or evaluations) [15]. The convex integral can be generalized by replacing the set function $\tau \circ \mu$ in its definition with an arbitrary monotonic set function ν on $\mathcal{P}(\Omega)$ such that $\nu(\varnothing) = -\infty$ and $\nu(\Omega) = 0$, also called a penalty on Ω [15].

In particular, the convex integral with respect to completely maxitive capacities (or penalties) is strictly related to the idempotent integral of tropical mathematics [7] and to convex measures of risk [24]. It corresponds to the functional $f \mapsto \bigvee_{\omega \in \Psi} (f(\omega) - \psi(\omega))$ on $\mathcal{E}$, where $\Psi \subseteq \Omega$ is not empty and ψ is a real-valued function on Ψ such that $\bigwedge_{\omega \in \Psi} \psi(\omega) = 0$.

5 Conclusion

The present paper studied maxitive integrals with respect to maxitive capacities, and in particular the Shilkret and convex integrals. These have particularly simple expressions when the capacities are completely maxitive. In this case, the Shilkret and convex integrals can be characterized as evaluations of uncertain payoffs by few basic decision-theoretic properties. These will be discussed in future work, with particular emphasis on the case of likelihood-based decision making [14,26].

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